

Compendium for MA3002 General topology

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1 Lexicon

Accumulation point, limit point, cluster point, point of accumulation, if A is a subset of a topological space X and x is a point of X , we say that x is an *accumulation point* of A if every neighbourhood of x (open sets containing x) intersects A in some other point other than x itself. In other words, x is an *accumulation point* of A if it belongs to the closure of $A - \{x\}$.

Example 1 On the real line $\mathbb{R}$ with the subset $A = (0, 1]$, every point on the interval $[0, 1]$ is an accumulation point of A .

Example 2 On the real line $\mathbb{R}$ with the subset $B = \{1/n | n \in \mathbb{Z}_+\}$, 0 is the only accumulation point of B . All other points in $\mathbb{R}$ has a neighbourhood that does not intersect B , or it intersects B only in the point x itself.

Example 3 On the real line $\mathbb{R}$ with the subset $C = \mathbb{Q}$, each point in $\mathbb{R}$ is an accumulation point of $\mathbb{Q}$.

Example 4 Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\{a, b\}, \{c, d\}\}$. If $A = \{a, b\}$, b is an accumulation point in A for a .

Ball, given a set X , a metric d and $\epsilon > 0$, the set

$$B_d(x, \epsilon) = \{y | d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ is called the ϵ -ball centered at x .

Basis, for a set X , a *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X (called *basis elements*) such that:

- (1) For each $x \in X$, there is at least one basis element B containing x .
- (2) if x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

Such a $\mathcal{B}$ will define a topology $\mathcal{T}$: A subset $U \subseteq X$ is an element of $\mathcal{T}$ (that is, an open set) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. Each element in $\mathcal{T}$ can be expressed as a union of basis elements (not necessarily uniquely, however).

Example 1 The collection $\mathcal{B}$ of all circular regions (open balls, circle interiors) in the plane is a basis.

Example 2 The collection $\mathcal{B}$ of all open rectangles (rectangle interiors) in the plane is a basis.

Boundary, the *boundary* of a subset A of a topological space X is defined as the points which do not belong to the interior or the exterior of A . It is denoted $b(A)$. It is not the same as the set of accumulation points. The set of accumulation points of A can contain both boundary and interior points from A .

Bounded, given a metric space X with metric d , a subset A of X is said to be *bounded* if there is some number M such that

$$d(a_1, a_2) \leq M$$

for every pair a_1, a_2 of points of A .

Box topology, let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let the basis for the topology on the product space

$$\prod_{\alpha \in J} X_\alpha,$$

be the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha,$$

where U_α here U_α is open in X_α for each $\alpha \in J$. The topology generated by this basis is called the *box topology*. The box topology is in general finer than the product topology.

Cauchy sequence, in a metric space (X, d) , a sequence $\{x_n\}$ of points in X is called a *Cauchy sequence* in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

Closed set, a subset A of a topological space X is said to be *closed* if the set $X - A$ is open. If Y is a subspace of X , a set A is *closed* in Y if A is a subset of Y and if A is closed in the subspace topology of Y (that is, $Y - A$ is open in Y). Theorem: A subset A of a topological space X is *closed* if and only if A contains each of its accumulation points.

Closure, the *closure* of a subset A of a topological space X is defined as the intersection of all closed sets containing A , or equivalently, the smallest closed set containing A . It is denoted $\overline{A}$.

Coarse/Fine, if $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is *finer* than $\mathcal{T}$. If the containment is proper, we say that $\mathcal{T}'$ is *strictly finer* than $\mathcal{T}$. Additionally, we say that $\mathcal{T}$ is *coarser* or *strictly coarser* than $\mathcal{T}'$ in these two situations. We say that $\mathcal{T}$ is *comparable* with $\mathcal{T}'$ if either $\mathcal{T} \subseteq \mathcal{T}'$ or $\mathcal{T}' \subseteq \mathcal{T}$.

Compact, a subset A of a topological space X is said to be *compact* if every open covering $\mathcal{A}$ of A contains a finite subcollection that also covers A .

Example 1 Every closed and bounded interval $[a, b]$ on $\mathbb{R}$ is compact (Heine-Borel theorem).

Example 2 $A = (0, 1)$ on $\mathbb{R}$ is not compact. An open cover is $\bigcup_{n=1}^{\infty} (\frac{1}{n+2}, \frac{1}{n})$, but contains no finite subcover.

Complete, a metric space (X, d) is complete if every Cauchy sequence converges.

Connected, a topological space X is *connected* if it is not possible to find a *separation* of X into two disjoint nonempty open subsets U, V whose union is X . Equivalently, a space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

Continuity, a function $f : X \mapsto Y$ between two topological spaces is *continuous* if for every open set $V \subseteq Y$, the inverse image $f^{-1}(V) = \{x \in X | f(x) \in V\}$ is also an open set (that is, the inverse image is an element in $\mathcal{U}$).

Continuity (pointwise), see theorem 18.1.

Convergence, an arbitrary sequence $x_1, x_2, \dots$ of points of a topological space X is said to *converge* to the point x of X if for each neighbourhood (open set) U around x , there is a positive integer N such that $x_n \in U$ for all $n \geq N$.

Example 1 In the discrete topology of the set $X = \{a, b, c\}$, the sequence $a, a, a, \dots$ converges to all of a, b and c . So there need not be just one convergence point.

Convex, given an ordered set X , a subset Y of X is *convex* in X if for each pair of points $a < b$ of Y , the entire interval (a, b) of points of X lies in Y .

Cover, a collection $\mathcal{A}$ of subsets of a topological space X is said to *cover* X , or to be a *covering* of X , if the union of the elements of $\mathcal{A}$ is equal to X . It is called an *open covering* of X if its elements are open subsets of X .

Dense, a subset A of a topological space X is said to be *dense* in X if $\overline{A} = X$.

Diameter, let X be a metric space with metric d , and let $A \subseteq X$. If A is nonempty and bounded (there exists M such that $d(x, y) \leq M$ for all $x, y \in A$), the *diameter* of A is defined as

$$\text{diam } A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Disconnected, A subset A of a topological space X is *disconnected* if there exist open subsets G, H of X such that $A \cap G$ and $A \cap H$ are disjoint non-empty sets whose union is A . $G \cup H$ is called a *disconnection* of A .

Discrete topology, the *discrete topology* of a set X is the power set of X . In this topology, all subsets of X are both open and closed.

Embedding, let $f : X \mapsto Y$ be an injective continuous map, where X, Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y , then the function $f' : X \mapsto Z$ obtained by restricting the range of f is bijective. If f' is homeomorphism of X with Z , then we say that the map $f : X \mapsto Y$ is a (*topological*) *embedding* of X in Y . Easier explanation: A topological space X is said to be *embedded* in a topological space Y if X is homeomorphic to a subspace of Y .

Euclidean metric, the *Euclidean metric* d on $\mathbb{R}^n$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = [(x_1 - y_1)^2 + \dots + (x_n - y_n)^2]^{1/2}.$$

Exterior, the *exterior* of a subset A of a topological space X is defined as the union of all open sets (or the largest open set) contained in A^c . It is denoted $\text{Ext } A$.

Finite complement topology

Hausdorff, a topological space $(X, \mathcal{U})$ is *Hausdorff* if for any two distinct points $x, y \in X$ there exist disjoint neighbourhoods (open sets) of x and y .

Homeomorphism, a function $f : X \mapsto Y$ between two topological spaces is called a *homeomorphism* if it has the following properties:

- f is a bijection
- f is continuous
- f^{-1} is continuous (f is an open mapping)

Interior, the *interior* of a subset A of a topological space X is defined as the union of all open sets contained in A . It is denoted $\text{Int } A$.

Interior point, a point $x \in \text{Int } A$ (the interior of the set A), where A is a subset of a topological space X .

Linear continuum, a simply ordered set L having more than one element is called a *linear continuum* if the following hold:

- (1) L has the least upper bound property
- (2) If $x < y$, there exists z such that $x < z < y$.

Locally compact

Metric, a metric on a set X is a function

$$d : X \times X \mapsto \mathbb{R}$$

having the following properties:

- (1) $d(x, y) \geq 0$ for all $x, y \in X$, $d(x, y) = 0$ holds if and only $x = y$.
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
- (3) (Triangle equality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

$d(x, y)$ is called the *distance* between x and y in the metric d .

Metric space, a *metric space* is a metrizable space X together with a specific metric d that gives the topology of X .

Metric topology, if d is a metric on the set X , then the collection of all ϵ -balls $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$ is a basis for a topology on X called the *metric topology* induced by d .

Example 1 For any set X , let $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Then d is a metric, and the topology it induces is the discrete topology.

Metrizable, a topological space X is said to be *metrizable* if there exists a metric d on the set X that induces the topology of X .

Neighbourhood, a *neighbourhood* around a point $p \in X$ is defined in the book as an open subset of X containing p . (Warning, other books have a different definition.)

Norm, given $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{R}^n$, we define the *norm* of $\mathbf{x}$ as

$$||x|| = (x_1^2 + \dots + x_n^2)^{1/2}.$$

Normal, given a topological space X , X is said to be *normal* if for each pair A, B of disjoint closed sets of X , there exist disjoint open sets containing A and B , respectively.

Open set, given a topological space $(X, \mathcal{U})$, the members of $\mathcal{U}$ are called *open sets*.

Order topology, let X be an ordered set with more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

- (1) All open intervals (a, b) in X .
- (2) If X has a minimal element a_0 , all intervals of the form $[a_0, b)$.

(3) If X has a maximal element b_0 , all intervals of the form $(a, b_0]$.

The collection $\mathcal{B}$ is a basis for the *order topology*.

Example 1 The intervals (a, b) for all $a < b$ forms a basis for the order topology on $\mathbb{R}$.

Path connected, given a topological space X and two points $x, y \in X$, a *path* in X from x to y is a continuous map $f : [a, b] \mapsto X$ of some closed interval in the real line into X , such that $f(a) = x$ and $f(b) = y$. A space X is said to be *path connected* if every pair of points of X can be joined by a path in X .

Example 1 TODO page 156 example 6 and 7.

Example 2 The *topologist's sine curve* defined by the graph of $\sin(1/x)$ over the interval $(0, 1]$ is connected, but not path connected, since the curve includes $(0, 0)$ which has no path to the function. Each point in $A = \{(0, y) | -1 \leq y \leq 1\}$ is an accumulation point of $B = \{(x, y) | y = \sin(1/x)\}$, so A and B are connected.

Product topology, let S_β denote the collection

$$S_\beta\{\pi_\beta^{-1}(U_\beta) | U_\beta \text{ open in } X_\beta\}$$

and let S denote the union of these collections,

$$S = \bigcup_{\beta \in J} S_\beta.$$

The topology generated by the subbasis S is called the *product topology* on the product space $\prod_{\alpha \in J} X_\alpha$. $\pi_\beta : \prod_{\alpha \in J} X_\alpha \mapsto X_\beta$ is the projection of an element in the product space into the β -th coordinate: $\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$.

A typical element of the basis for a product topology can be described as follows: Let $\beta_1, \dots, \beta_n$ be a finite set of distinct indices from the index set J , and let U_{β_i} be an open set in X_{β_i} for $i = 1, \dots, n$. Then

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \dots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

is a typical element in the basis. There is no restriction whatsoever on the α -th coordinate of x if α is not one of the indices β_i .

Projection, let $X \times Y$ be a product topology and let $\pi_1 : X \times Y \mapsto X$ be defined by

$$\pi_1(x, y) = x$$

and let $\pi_2 : X \times Y \mapsto Y$ be defined by

$$\pi_2(x, y) = y.$$

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

Quotient map, let X and Y be topological spaces, let $p : X \mapsto Y$ be a surjective map. The map p is said to be a *quotient map* provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X .

Ray topology, if X is an ordered set, and a is an element of X , there are four subsets of X that are called the *rays* determined by a . They are the following:

$$\begin{aligned}(a, +\infty) &= \{x | x > a\}, \\ (-\infty, a) &= \{x | x < a\}, \\ [a, +\infty) &= \{x | x \geq a\}, \\ (-\infty, a] &= \{x | x \leq a\}.\end{aligned}$$

Sets of the first two types are called *open rays*, and sets of the last two types are called *closed rays*.

Regular, a topological space X is *regular* if $F \subseteq X$ is closed and $p \in X$ does not belong to F and there exists disjoint open sets G, H such that $F \subseteq G$ and $p \in H$.

Separable, a topological space X is *separable* if it has a countable dense subset.

Separated, two subsets A, B of a topological space X are said to be *separated* if A, B are disjoint and neither contains an accumulation point of the other.

Separation, given a topological space X , a *separation* of X is a pair U, V of disjoint open subsets of X whose union is X .

Separation axioms, let X be a topological space. X is *Hausdorff* if for every pair $x, y \in X$, there are disjoint open sets U, V such that $x \in U$ and $y \in V$.

X is *regular* if for each $x \in X$ and closed subset $B \subseteq X$ such that $x \notin B$, there exist disjoint open sets such that $x \in U$ and $B \subseteq V$.

X is *normal* if for each pair of disjoint closed sets $A, B \subseteq X$, there exist disjoint open sets such that $A \subseteq U$ and $B \subseteq V$.

Square metric, the *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(\mathbf{x}, \mathbf{y}) = \max(|x_1 - y_1|, \dots, |x_n - y_n|).$$

Standard bounded metric, let X be a metric space with metric d . Define $\bar{d} : X \times X \mapsto \mathbb{R}$ as

$$\bar{d}(x, y) = \min(d(x, y), 1).$$

$\bar{d}$ is called the *standard bounded metric* corresponding to d and $\bar{d}$ is a metric that induces the same topology as d .

Standard topology, let $\mathcal{B}$ be the collection of all open intervals in the real line,

$$(a, b) = \{x | a < x < b\}.$$

The topology generated by $\mathcal{B}$ is called the *standard topology* on $\mathbb{R}$.

Subbasis, a collection $\mathcal{S}$ of subsets of X whose union is X is a *subbasis* for X , and generates a topology $\mathcal{T}$ by taking all unions of finite intersections of elements in $\mathcal{S}$.

Subspace topology, let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X , the collection

$$\mathcal{T}_Y = \{Y \cap U | U \in \mathcal{T}\}$$

is a topology on Y , called the *subspace topology*. With this topology. Y is called a *subspace* of X ; its open sets consist of all intersection of open sets of X with Y . See Lemma 16.1 for basis.

Example 1 See page 90 for two examples (2 and 3) where X is an ordered set with the order topology, and that an open set in a subset $Y \subseteq X$ need not be open in X .

Topological space, a topological space $(X, \mathcal{U})$ is a set X together with a collection $\mathcal{U}$ of subsets of X , satisfying the following axioms:

- $\emptyset, X \in \mathcal{U}$
- $\mathcal{U}$ is closed under union of any number (possibly not finite) of elements
- $\mathcal{U}$ is closed under intersection of any two elements

$\mathcal{U}$ is called a *topology* on X .

Example 1 The closed intervals of $\mathbb{R}$ is not a topology set on $\mathbb{R}$. The union of $\{[-n, n] | n \in \mathbb{N}\} = (-\infty, \infty)$ is not a closed interval.

Example 2 If X is any set, the power set of X is a topology on X and is called the *discrete topology* on X .

Example 3 If X is any set, the collection consisting of X and $\emptyset$ is a topology on X and is called the *trivial topology* or *indiscrete topology*.

Uniform convergence, let $f_n : X \mapsto Y$ be a sequence of functions from the set X to the metric space Y . Let d be the metric for Y . We say that the sequence (f_n) *converges uniformly* to the function $f : X \mapsto Y$ if given $\epsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all x in X .

Uniform metric, given an index set J , and given points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ on $\mathbb{R}^J$, define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup(\bar{d}(x_\alpha, y_\alpha) | \alpha \in J),$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. $\bar{\rho}$ is a metric and is called the *uniform metric* on $\mathbb{R}^J$ and the topology it induces is called the *uniform topology*.

2 Theorems

Lemma 13.1 Let X be a set, and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X . Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof See page 80.

Lemma 13.2 Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X .

Proof See page 80.

Theorem 15.1 If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y , then the collection

$$\mathcal{D} = \{B \times C | B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

Proof See page 87.

Theorem 15.2 The collection

$$\mathcal{S} = \{\pi_1^{-1}(U) | U \text{ open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof See page 88.

Lemma 16.1 If $\mathcal{B}$ is a basis for the topology of X , then the collection

$$\mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y .

Proof See page 89.

Lemma 16.2 Let Y be a subspace of X . If U is open in Y and Y is open in X , then U is open in X .

Proof Since U is open in Y , $U = Y \cap V$ for some set V open in X . Since Y and V are both open in X , so is $Y \cap V$.

Theorem 16.3 If A is a subspace of X and B is a subspace of Y , then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Proof See page 89-90.

Theorem 16.4 Let X be an ordered set in the order topology and let Y be a subset of X that is convex in X . Then the order topology on Y is the same as the topology Y inherits as a subspace of X .

Proof See page 91.

Theorem 17.1 Let X be a topological space. Then the following conditions hold:

- (1) $\emptyset$ and X are closed.
- (2) Arbitrary intersections of closed sets are closed.
- (3) Finite unions of closed sets are closed.

Proof See page 94.

Theorem 17.2 Let Y be a subspace of X . Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y .

Proof See page 94.

Theorem 17.3 Let Y be a subspace of X . If A is closed in Y and Y is closed in X , then A is closed in X .

Proof The proof is kindly left for “you”.

Theorem 17.4 Let Y be a subspace of X and let A be a subset of Y , let $\overline{A}$ denote the closure of A in X . Then the closure of A in Y equals $\overline{A} \cap Y$.

Proof See page 95.

Theorem 17.5 Let A be a subset of the topological space X .

- (a) Then $x \in \overline{A}$ if and only if every open set U containing x intersects A .
- (b) Supposing the topology of X is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A .

Proof See page 96.

Theorem 17.6 Let A be a subset of a topological space X , let A' be the set of all accumulation points of A . Then

$$\overline{A} = A \cup A'.$$

Proof See page 97.

Corollary 17.7 A subset of a topological space is closed if and only if it contains all its accumulation points.

Theorem 17.8 Every finite point set in a Hausdorff space X is closed.

Proof See page 99.

Theorem 17.10 If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X .

Proof Suppose that x_n is a sequence of points of X that converges to x . If $y \neq x$, let U and V be disjoint neighbourhoods of x and y , respectively. Since U contains x_n for all but finitely many values of n , the set V cannot. Therefore, x_n cannot converge to y .

Theorem 17.11 Every simple ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.

Theorem 18.1 Let X and Y be topological spaces and let $f : X \mapsto Y$. Then the following are equivalent:

- (1) f is continuous.
- (2) For every subset A of X , one has $f(\overline{A}) \subseteq \overline{f(A)}$.
- (3) For every closed set B of Y , the set $f^{-1}(B)$ is closed in X .
- (4) For each $x \in X$ and each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$.

If condition (4) holds for the point x in X , we say that f is *continuous at the point x* .

Proof See page 104.

Theorem 18.2 Rules for constructing continuous functions, see book page 107-108.

Theorem 19.1 (Comparison of the box and product topologies) The box topology on $\prod X_\alpha$ has a basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α . The product topology on $\prod X_\alpha$ has a basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and U_α equals X_α except for finitely many values of α .

Theorem 24.1 If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L .

3 Counterexamples, corner cases and similar stuff

Separation There exist disjoint sets A, B with elements $x \in A, y \in B$ such that $d(x, y) = 0$:
 $X = \mathbb{R}^2$ with Euclidean norm, $A = \{(x, y) | xy \geq -1, x < 0\}, B = \{(x, y) | xy \geq 1, x > 0\}$.

4 Exam 2008

Problem 1 Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathcal{T} = \{\emptyset, E_n\}$ where $E_n = \{n, n+1, \dots\}, n \in \mathbb{N}$.

- a) Show that $\mathcal{T}$ is a topology on $\mathbb{N}$.
- b) Find all the accumulation points of $\{4, 13, 28, 37\}$.
- c) Find the closure of $\{7, 24, 47, 85\}$ and $\{3, 6, 9, 12, \dots\}$.
- d) Find the dense subsets of $\mathbb{N}$.

Solution: a) Axiom 1: $\emptyset \in \mathcal{T}$ is given in the text. $E_n \in \mathcal{T}$ for all $n \in \mathbb{N}$, hence $E_1 = \mathbb{N} \in \mathcal{T}$.

Axiom 2: $E_a \cup E_b = E_{\min(a,b)} \in \mathcal{T}$. Similar argument for arbitrary unions.

Axiom 3: $E_a \cap E_b = E_{\max(a,b)} \in \mathcal{T}$. Hence, $\mathcal{T}$ is a topology on $\mathbb{N}$.

b) A point x is an accumulation point of A if it belongs to the closure of $A - \{x\}$. The set of accumulation points is $\{1, 2, 3, \dots, 36\}$. Equivalent explanation: A point x is an accumulation point of A if every open set containing it also contains another point from A . All open subsets containing $x \leq 36$ must also contain $37 \in A$. This is not true for $x = 37$, since the open set $\{37, 38, \dots\}$ does not contain any other points from A .

c) First off, the closed sets in $\mathcal{T}$ are $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, \dots, n\}, \dots\}$. The closure of a set A is the intersection of all closed sets containing A . Hence, the closure of $\{7, 24, 47, 85\}$ is $\{1, 2, 3, \dots, 85\}$, and the closure of $\{3, 6, 9, 12, \dots\}$ is $\mathbb{N}$.

d) My answer: The dense subsets of $\mathbb{N}$ are of the form $\mathbb{N} - A$ where A is finite. Alternate answer: If a subset A of $\mathbb{N}$ is infinite, or equivalently unbounded, then $\overline{A} = \mathbb{N}$. If A is finite, then its closure is not equal to $\mathbb{N}$.

Problem 2 Let A and B be subsets of a topological space X .

- a) Show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Give an example which shows that equality doesn't hold.
- b) Let $\{X_i\}_{i \in I}$ be an indexed family of topological spaces. Let $A_i \subseteq X_i$ for all i . Let $X = \prod_{i \in I} X_i$. X can have either the product topology or the box topology. Show that in both cases,

$$\prod \overline{A_i} = \overline{\prod A_i}.$$

Solution: a) (Munkres 17.8 a)): First: If $A \subseteq B$, then all limit points of A are also limit points of B , so $\overline{A} \subseteq \overline{B}$. Then, $\overline{A \cap B} \subseteq \overline{A}$ and $\overline{A \cap B} \subseteq \overline{B}$, so $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. A counterexample to equality is $A = [0, 1), B = [1, 2]$ in $\mathbb{R}$: $\overline{A \cap B} = \emptyset$ and $\overline{A} \cap \overline{B} = \overline{[0, 1)} \cap \overline{[1, 2]} = [0, 1] \cap [1, 2] = \{1\}$.

b) (Theorem 19.5 in Munkres): Let $\mathbf{x} = \{x_\alpha\}$ be a point of $\prod \overline{A_\alpha}$. Show that $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element for either the box or product topology that contains $\mathbf{x}$. Since $x_\alpha \in \overline{A_\alpha}$, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$ for each α . Then $\mathbf{y} = \{y_\alpha\}$ belongs to both U and $\prod A_\alpha$. Since U is arbitrary, it follows that $\mathbf{x}$ belongs to the closure of $\prod A_\alpha$.

Conversely, suppose $\mathbf{x} = \{x_\alpha\}$ lies in the closure of $\prod A_\alpha$ in either topology. We show that for any given index β , we have $x_\beta \in \overline{A_\beta}$. Let V_β be an arbitrary open set of X_β containing x_β . Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, it contains a point $\mathbf{y} = \{y_\alpha\}$ of $\prod A_\alpha$. Then y_β belongs to $V_\beta \cap A_\beta$. It follows that $x_\beta \in \overline{A_\beta}$.

Problem 3

- a) Define the terms *connected* and *path connected*. What's the connection between them?
- b) Let A be a connected subspace (subset) of a topological space X . Show that if

$$A \subseteq B \subseteq \overline{A} \subseteq X,$$

then B is connected.

Solution: a) A topological space X is connected if it is not possible to find a separation of X into two disjoint nonempty open subsets U, V whose union is X .

A topological space X is path connected if for every pair of points $x, y \in X$ can be joined by a path. A path in X from x to y is a continuous map $f : [x, y] \mapsto X$ (from a closed interval on the real line) such that $f(a) = x$ and $f(b) = y$.

Connection: If a space X is path connected $\Rightarrow X$ is connected. (However, the other direction is not true in general.)

- b) (Theorem 23.4, Munkres p.150) Different statement: If A is connected, and B is formed by adding to A some (or all) of its accumulation points, then B is also connected.

Proof: Let A be connected, and let $A \subseteq B \subseteq \overline{A}$. Suppose that $B = C \cup D$ is a separation of B . Then by (Lemma 23.2 in Munkres), A must lie entirely in C or D . Suppose (without loss of generality) that $A \subseteq C$. Then $\overline{A} \subseteq \overline{C}$. Since $\overline{C}$ and D are disjoint, B cannot intersect D . This contradicts the fact that D is a nonempty subset of B . Hence, B is connected since there is no separation of B .

Problem 4 Let $f, g : X \mapsto Y$ be two continuous functions between two topological spaces X and Y , where Y is a Hausdorff space. Show that

$$\{x | f(x) = g(x)\}$$

is closed in X .

Solution: TODO

Problem 5 Show the following:

- a) Every closed subset of a compact space is compact.
- b) Every compact subset of a Hausdorff space is closed.
- c) Every compact Hausdorff space is regular.

Solution a) (Theorem 11.3 in Schaum) Let F be an arbitrary closed subset of a compact space X . Let $\mathcal{G} = \{G_i\}$ be an open cover of F , that is, $F \subseteq \cup_i G_i$. Then $X = (\cup_i G_i) \cup F^c$, that is, $\mathcal{G}^* = \{G_i\} \cup \{F^c\}$ is a cover of X . But F^c is open since F is closed, so $\mathcal{G}^*$ is an open cover of X . By hypothesis, X is compact, hence $\mathcal{G}^*$ is reducible to a finite cover of X , say,

$$X = G_{i_1} \cup \dots \cup G_{i_m} \cup F^c, G_{i_k} \in \mathcal{G}.$$

But F and F^c are disjoint, hence

$$F \subseteq G_{i_1} \cup \dots \cup G_{i_m}, G_{i_k} \in \mathcal{G}.$$

Hence, any open cover of F contains a finite subcover, hence F is compact.

b) (Theorem 11.5 in Schaum) Prove the equivalent, that if A is a compact subset of a Hausdorff space X , then A^c is open. Let $p \in A^c$. Then by problem 11.8 in Schaum, there exists an open set G_p such that $p \in G_p \subseteq A^c$. Hence, $A^c = \cup\{G_p | p \in A^c\}$. Hence, A^c is open as it is the union of open sets, and hence A is closed.

c) (Math3070) Let $F \subseteq X$ be closed and $x \notin F$. For every point $y \in F$, we have $x \neq y$. Since X is Hausdorff, there are open sets $U_y, V_y \subseteq X$ such that $U_y \cap V_y = \emptyset$ and $x \in U_y$ and $y \in V_y$. From a) and b), F is closed and compact. Then, we have found the desired $x \in X$ and $F \subseteq X$ and X is henceforth regular.

Problem 6 Are the following subsets of $\mathbb{R}$ compact?

i) $A = \{0\} \cup \{\frac{1}{n} | n \in \{1, 2, 3, \dots\}\}$

ii) $B = (0, 1]$.

Solution: a) (Example 26.2 in Munkres p.164) Given an open covering $\mathcal{A}$ of X , there is an element U of $\mathcal{A}$ containing 0. The set U contains all but finitely many of the points $1/n$. Choose, for each point X not in U , an element of $\mathcal{A}$ containing it. The collection consisting of these elements of $\mathcal{A}$, along with the element U , is a finite subcollection of $\mathcal{A}$ that covers X .

b) (Example 26.3 in Munkres p.164, Example 11.2.3 in Schaum p.152) Not compact. The covering $\mathcal{A} = (1/2, 1] \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+2}, \frac{1}{n})$ has no finite subcover.

Problem 7 Let $\mathbb{Q}$ denote the rational numbers as a subset of the real numbers $\mathbb{R}$.

a) Let $p \in \mathbb{Q}$, show that $\mathbb{R} - \{p\}$ is dense in $\mathbb{R}$.

b) Show that the set of irrational numbers $\mathbb{R} - \mathbb{Q}$ can be written as a countable union of open dense subsets in $\mathbb{R}$.

Solution: a) Let $q \in \mathbb{Q}$. Let q be an accumulation point for $\mathbb{R} - \{q\}$. Then, any neighbourhood around q , for instance $(q - 2\epsilon, q + 2\epsilon)$ for $\epsilon > 0$ contains the element $q - \epsilon$, which is different from q . Hence, $\mathbb{R} - \{q\}$ is dense in $\mathbb{R}$.

b) $\mathbb{R} - \mathbb{Q} = \bigcup_{q \in \mathbb{Q}} (\mathbb{R} - \{q\})$. The number of q 's are countable, and $\mathbb{R} - \{q\}$ is dense in $\mathbb{R}$ according to a).

5 Eksamen 2009

Problem 1 A topological space $(X, \mathcal{T})$ is given by:

$$X = \{a, b, c, d, e\}$$

$$\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$$

a) Determine the closed subsets of X .

b) Determine the closure of the subsets $\{a\}$, $\{b\}$ and $\{c, e\}$.

c) Determine the dense subsets of X .

Solution: a) The closed subsets are $\emptyset, X, \{b, c, d, e\}, \{c, d, e\}, \{b, e\}, \{e\}, \{c, d\}$.

b) The closure of a set A is the smallest closed set containing A . So the solution is $\overline{\{a\}} = X, \overline{\{b\}} = \{b, e\}, \overline{\{c, e\}} = \{c, d, e\}$.

c) A subset $A \subseteq X$ is dense if its closure is X . The dense subsets of X are the 16 subsets containing a . $\{b, c, d, e\}$ is a closed subset, and is a superset of all subsets that don't contain a .

Problem 2 Let A, B, A_α (α is an element of an arbitrary index set) be subsets of a topological space X .

a) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b) Show that

$$\overline{\bigcup A_\alpha} \supseteq \bigcup \overline{A_\alpha}.$$

Give an example to show that equality doesn't hold.

Solution: a) We have $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$, because $A \subseteq A \cup B$ and $\overline{A \cup B}$ is a closed set which contains $\overline{A}$. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. An open set is a subset of its closure: $A \cup B \subseteq \overline{A \cup B}$. $\overline{A \cup B}$ is a closed set since it is the union of two closed sets (since it is equivalent to taking the intersection of their complements (open sets), see theorem 17.1 in Munkres). Also, if $\overline{A}$ is the closure of A and F is a closed superset of A , then $A \subseteq \overline{A} \subseteq F$ (since $\overline{A}$ is the intersection of all closed sets containing A , or equivalently, the smallest closed set containing A). Hence, $A \cup B \subseteq \overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ and therefore $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b) $A_\alpha \subseteq \bigcup_\alpha A_\alpha \subseteq \overline{\bigcup_\alpha A_\alpha}$ which is a closed set, hence $\overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$. This holds for all α , so $\bigcup_\alpha \overline{A_\alpha} \subseteq \overline{\bigcup_\alpha A_\alpha}$. Let $A_r = \{r\}, r \in \mathbb{Q}$. Then $\overline{A_r} = \{r\}$ and $\bigcup_{r \in \mathbb{Q}} \overline{A_r} = \mathbb{Q}$ whereas $\overline{\bigcup_{r \in \mathbb{Q}} A_r} = \overline{\mathbb{Q}} = \mathbb{R}$.

Problem 3 Let $\mathbb{R}$ be the real numbers with the finite complement topology, that is U is open if $\mathbb{R} - U$ is finite, or $\mathbb{R} - U = \mathbb{R}$. Show that this is a topology on $\mathbb{R}$.

Consider the sequence $\{\frac{1}{n}\}_{n=1,2,\dots}$. Does the sequence converge? If yes, towards what?

Solution: Denote the topology by $\mathcal{T}$ and let $X = \mathbb{R}$.

(i) Both X and $\emptyset$ are in $\mathcal{T}$, since $X - X$ is finite and $X - \emptyset$ is all of X .

(ii) If $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}$,

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha),$$

the latter set is finite because each set $X - U_\alpha$ is finite.

(iii) If $U_1, \dots, U_n$ are nonempty elements of $\mathcal{T}$,

$$X - \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X - U_i),$$

the latter set is a finite union of finite sets, and is therefore finite.

The sequence converges to every real number, due to the following theorem! Let X be a set equipped with the finite complement topology. If $(x_n)_{n \in \mathbb{Z}_+}$ is an infinite sequence of distinct points in X then (x_n) converges to every x in X . Proof: Let U be a neighbourhood of $x \in X$,

i.e. let $X - U$ be finite. It follows that $x_n \in U$ for all, but finitely many $n \in \mathbb{Z}_+$, i.e. (x_n) converges to x .

Problem 4 Show that a topological space X is Hausdorff if and only if the diagonal Δ is closed in $X \times X$.

Solution: $\Rightarrow$: Suppose X is Hausdorff. The diagonal Δ is closed if and only if the complement $\Delta^c = X \times X - \Delta$ is open. Let $(a, b) \in \Delta^c$, i.e. a and b are distinct points in X . Since X is Hausdorff there exists open sets U and V in X such that $a \in U, b \in V$ and $U \cap V = \emptyset$. Hence $(a, b) \in U \times V$ and $U \times V$ is open in $X \times X$. Furthermore, $(U \times V) \cap \Delta = \emptyset$, since U and V are disjoint. So for every point $(a, b) \in \Delta^c$ there exists an open set $U_{(a,b)}$ such that $(a, b) \in U_{(a,b)} \subseteq \Delta^c$. It follows (by exercise 13.1 in Munkres, bleh) that Δ^c is open, hence Δ is closed.

$\Leftarrow$: Suppose Δ is closed. If a and b are two distinct points in X then $(a, b) \in \Delta^c$. Since Δ^c is open there exists a basis element $U \times V$, U and V open in X for the product topology such that $(a, b) \in U \times V \subseteq \Delta^c$. Since $U \times V \subseteq \Delta^c$ it follows that $U \cap V = \emptyset$. Hence U and V are open sets such that $a \in U, b \in V$ and $U \cap V = \emptyset$, i.e. X is Hausdorff.

Problem 5

a) Let G be an open subset of $\mathbb{R}^2$. Show that if G is connected, G is also path connected.

b) Is $\mathbb{Q} \subseteq \mathbb{R}$ a connected subset?

Solution: a) (Munkres 24.10, p.158) Hint: Show that given $x_0 \in U$, the set of points that can be joined to x_0 by a path in U is both open and closed in U .

Let U be a nonempty, open and connected subspace of $\mathbb{R}^2$ and let $x_0 \in U$. Furthermore, let A be the set of points in U that can be joined to x_0 by a path in U .

Check that A is open: Let $a_0 \in A \subseteq U$. Since U is open there is an open rectangle $V = (a, b) \times (c, d)$ such that $a_0 \in V \subseteq U$. Since V clearly is path connected it follows that $V \subseteq A$, hence A is open.

Check that A is closed: let $u_0 \in U - A$. If every open rectangle containing u_0 intersects A , then clearly $u_0 \in A$, hence u_0 is not a limit point of A . Thus no point of $U - A$ is a limit point of A , hence A is closed.

Since U is connected it follows that $A = U$, hence U is path connected.

b) No, $\mathbb{Q}$ is disconnected everywhere in $\mathbb{R}$. Let $a, b \in \mathbb{Q}$ with $a < b$. Then there exists an irrational number c such that $a < c < b$. Then $(-\infty, c)$ and (c, ∞) (as subsets of $\mathbb{Q}$) is a separation of $\mathbb{Q}$ into two non-empty open subsets such that their union is $\mathbb{Q}$, and hence $\mathbb{Q}$ is disconnected. (In fact, the only connected subsets of $\mathbb{Q}$ are the singletons.)

Problem 6 Let X and Y be two connected topological spaces. Show that $X \times Y$ is connected.

Solution: Let $p = (x_1, y_1)$ and $q = (x_2, y_2)$ be a pair of points in $X \times Y$. Now $\{x_1\} \times Y$ is homeomorphic to Y and is therefore connected. Similarly, $X \times \{y_2\}$ is connected.

But $\{x_1\} \times Y \cap X \times \{y_2\} = \{(x_1, y_2)\}$, hence $\{x_1\} \times Y \cup X \times \{y_2\}$ is connected. Accordingly, p and q belong to the same component. But p and q were arbitrary, hence $X \times Y$ has one component and is therefore connected.

Problem 7 Let X be a topological space and $f : X \mapsto \mathbb{R}$ a continuous function. Show that if

X is compact, then there exist c, d such that for every $x \in X$,

$$f(c) \leq f(x) \leq f(d).$$

Solution: Apply theorem 27.4 (Extreme value theorem (Munkres, p.174). Outline:

f continuous, X compact $\Rightarrow A = f(X)$ is compact.

Assume A has no largest element, then $\{(-\infty, a) | a \in A\}$ is an open covering of A . Since A is compact, some finite subcollection $\{(-\infty, a_i), \dots\}$ covers A . But then the largest element a_j belongs to none of these sets, and hence there is a largest element. Same argumentation for smallest, and hence A has a smallest element also.

Problem 8 Let $X = \mathbb{R}^\infty = \prod_{i=1}^\infty \mathbb{R}_i$ where $\mathbb{R}_i = \mathbb{R}$, the real numbers with standard topology. Define $f : \mathbb{R} \mapsto \mathbb{R}^\infty$ by

$$f(t) = (t, t, t, \dots).$$

Is f continuous in the product topology and the box topology?

Solution If f is continuous, then $f^{-1}(U)$ is open for every open $U \in \mathbb{R}^\infty$.

Box topology: We will try to find an open set O in $\mathbb{R}^\infty$ such that $f^{-1}(O)$ isn't open. Consider $O = (I_1, I_2, \dots)$ where $I_n = (-\frac{1}{n}, \frac{1}{n})$. Then, $f^{-1}(O) = \{0\}$ is the preimage. However, it is closed, and hence f is not continuous in the box topology.

An open set $A = \prod A_\alpha$ in the product topology space has a finite number of coordinates where $A_i \neq \mathbb{R}$. Hence, $f^{-1}(A) = \bigcap A_i$ where the right side is a finite intersection of open sets, hence $f^{-1}(A)$ is open, hence f is continuous in the product topology.

6 References

- Topology (James Munkres), second edition
- Schaum's outline of general topology
- <http://www.math.ku.dk/~moller/e03/3gt/3gt.html>
- <http://www.math.cuhk.edu.hk/course/math3070/>