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Coupling constants and the generalized Riemann problem for isothermal junction flow

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Abstract. We consider gas flow in pipe networks governed by the isothermal Euler equations. A set of coupling conditions is required to completely specify the Riemann problem at the junction. The momentum-related condition has no obvious expression and different approaches have been used in previous work. For the condition of equal momentum flux, Colombo and Garavello [A well posed Riemann problem for the p-system at a junction, *Netw. Heterog. Media* **1** (2006) 495–511] proved existence and uniqueness of solutions globally in time and locally in the subsonic region of the state space. If the entropy constraint is not considered, we are able to prove existence and uniqueness globally in the subsonic region for any momentum-related coupling constant satisfying a monotonicity requirement. The previously suggested conditions of equal pressure and equal momentum flux satisfy this requirement, but in general they both fail to fulfill the entropy constraint. The classical Bernoulli invariant is a natural scalar formulation of momentum conservation under ideal flow conditions. Our analysis shows that this invariant is monotone and unconditionally leads to solutions satisfying the entropy constraint. Of the coupling constants considered, this is therefore the only choice that guarantees the unique existence of *entropy* solutions to the N-junction Riemann problem for all initial data in the subsonic region.

Keywords: Gas flow; networks; junctions.

Mathematics Subject Classification 2010: 35L65, 76N15

1. Introduction

This paper is concerned with a particular instance of a more general question; how to properly define global weak solutions for hyperbolic conservation laws defined on N segments of the real line, connected by a junction. Such conservation laws are given by

$$\frac{\partial \boldsymbol{U}_i}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}_i) = 0, \quad i \in \{1, \dots, N\}, \tag{1.1}$$

where in each segment i, we seek the solution $\boldsymbol{U}_i(x,t)$ for

$$t \in \mathbb{R}^+, \tag{1.2}$$

$$x \in \mathbb{R}^+. \tag{1.3}$$

The segments are assumed to be connected at the origin, as schematically illustrated in Fig. 1.

Herein, for any segment i we may instead of (1.3) consider a finite interval $x \in (0, b_i)$ if proper boundary conditions may be supplied at $x = b_i$.

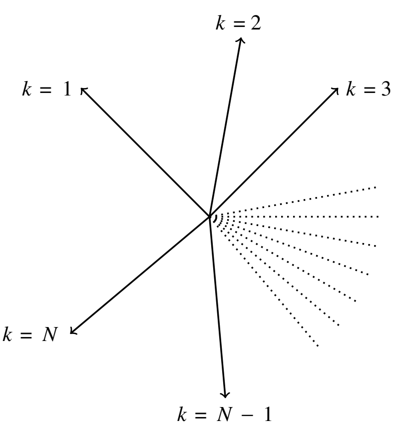

Fig. 1. An N-junction. The different segments are joined at a vertex, with the positive x-direction always pointing away from the junction.

We observe that even in the scalar case, the initial value problem for (1.1) given by

$$\boldsymbol{U}_i(x,0) = \boldsymbol{U}_{i,0}(x) \quad \forall\, i \in \{1,\ldots,N\} \tag{1.4}$$

is in general incompletely specified; boundary conditions, or *coupling conditions*, must be provided at the point $x = 0$ for all segments. The specification of such junction coupling conditions for the isothermal Euler equations of gas dynamics is the topic to be addressed in this paper. In the following, we will provide some background for this currently active field of research, somewhat related to the recent study of the coupling between different hyperbolic models at a fixed interface [1–3].

Our approach is similar in spirit to the work of Goudiaby and Kreiss [18] who considered open channel flow. The reader may also refer to the recent review article of Bressan *et al.* [9].

1.1. *The Riemann problem generalized to junctions*

Problems in the form (1.1)–(1.4) naturally arise in the study of traffic flow [10, 16, 21] and fluid flow in pipe networks [5–7, 11, 14, 19]. Central to the study of the well-posedness of any such model formulation is the concept of the N*-junction Riemann problem* [11, 14, 21], which may be stated as follows: Equations (1.1)–(1.3) are to be solved given constant initial data in each segment:

$$\boldsymbol{U}_i(x,0) = \bar{\boldsymbol{U}}_i \quad \forall\, i \in \{1,\ldots,N\}. \tag{1.5}$$

In general, one must expect that the evolved solutions $\boldsymbol{U}_i(x,t)$ depend on *all* initial states, $\bar{\boldsymbol{U}}_i$, through their interaction in the junction. One may however introduce a natural condition: in each segment, the solution should be compatible with a *standard* Riemann problem at the segment-junction interface [11, 14, 16]. This condition may be precisely stated as follows.

C1: For all $i \in \{1,\ldots,N\}$, there exists a state

$$\boldsymbol{U}_i^*(\bar{\boldsymbol{U}}_1,\ldots,\bar{\boldsymbol{U}}_N) = \lim_{x\to 0^+} \boldsymbol{U}_i(x,t) \tag{1.6}$$

such that $\boldsymbol{U}_i(x,t)$ is given by the restriction to $x \in \mathbb{R}^+$ of the Lax solution to the standard Riemann problem for $x \in \mathbb{R}$:

$$\begin{gathered} \frac{\partial \boldsymbol{U}_i}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}_i) = 0, \\ \boldsymbol{U}_i(x,0) = \begin{cases} \bar{\boldsymbol{U}}_i & \text{if } x > 0, \\ \boldsymbol{U}_i^* & \text{if } x < 0. \end{cases} \end{gathered} \tag{1.7}$$

In other words, $\boldsymbol{U}_i^*$ is the *similarity solution* $\boldsymbol{w}(x/t)$ to the Riemann problem (1.7) evaluated at $x/t = 0$. This concept of a "half-Riemann problem" was first considered by Dubois and LeFloch [15], and the Riemann problem for networks was first studied by Holden and Risebro [21].

To close the system, a number of additional *coupling conditions* are needed to relate the various vectors $\boldsymbol{U}_i^*$. These conditions should respect the following somewhat related considerations.

(i) The conditions should adequately represent the underlying physics we seek to describe by the model.
(ii) The conditions should, in conjunction with C1, lead to a well-posed initial value problem.

Arguably, (ii) could be considered a necessary requirement for (i).

1.2. *The isothermal Euler equations*

In this work, we follow the approach of [5, 6, 11, 20, 24, 25] and consider one-dimensional pipe flow governed by the *isothermal* Euler equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{1.8}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + p(\rho)) = 0. \tag{1.9}$$

Here, ρv is the mass flux, ρ is the fluid density and v is the fluid velocity. The isothermal equations assume the simplified pressure law:

$$p(\rho) = a^2 \rho, \tag{1.10}$$

where a is the constant fluid speed of sound. A more general formulation of (1.10) gives rise to the isentropic Euler equations considered in [11].

1.3. *Coupling conditions used with the isothermal Euler equations*

Two different coupling conditions [5, 6, 20] together with an entropy constraint [11] may be used to completely specify the problem. The first coupling condition is related to Eq. (1.8) and accounts for the conservation of mass at the junction. As remarked in [11], this is an obvious requirement. For N pipes of equal cross-section connected at a junction this may be stated as:

$$\sum_{i=1}^{N} \rho_i^*(x,t) v_i^*(x,t) = 0 \quad \text{for all } t > 0, \tag{1.11}$$

where in the context of (1.6) we have

$$\boldsymbol{U}_i^* = \begin{bmatrix} \rho_i^* \\ \rho_i^* v_i^* \end{bmatrix}. \tag{1.12}$$

Colombo and Garavello [11] proposed that an *entropy selection* principle should apply to solutions through the junction, analogous to the standard admissibility theory for weak solutions to conservation laws. A number of viable entropy–entropy flux pairs may be constructed for the one-dimensional equations (1.8) and (1.9) [23].

Garavello and Piccoli [17] note that for junctions, different entropies do not necessarily select the same solutions.

For isothermal flow, Colombo and Garavello [11] suggested using the *mechanical energy* as the entropy function. We will follow this approach as described in Sec. 3.

The final coupling condition is related to the momentum equations, (1.9), and does not seem to have an obvious expression. Colombo and Mauri [14] observe that a system described by the full set of Euler equations can in general not conserve linear momentum at the junction. On the contrary, the total momentum vector is constrained by the relative position of the pipes. For various flow models, momentum conservation has been replaced with the condition that some scalar flow parameter, $\tilde{\mathcal{H}}$, remains constant through the junction [6–8, 11, 14, 19, 20]. We will refer to such scalar parameters as momentum-related coupling *constants*.

In the recent literature, two approaches are seen to be the most common. These are the conditions of equal pressure [6, 11, 19, 20]:

$$p(\rho_i^*(x,t)) = \tilde{\mathcal{H}}_p \quad \text{for all } i \text{ and } t > 0, \tag{1.13}$$

and equal momentum flux [7, 8, 11, 14]:

$$(\rho_i^* v_i^{*2} + p(\rho_i^*))(x,t) = \tilde{\mathcal{H}}_{\mathrm{MF}} \quad \text{for all } i \text{ and } t > 0. \tag{1.14}$$

The choice of equal pressure is made primarily as it is a simple model that is widely used in the engineering community [6, 22, 24, 25]. The model is expected to be a fair approximation for low Mach number flows.

Colombo and Garavello [11] introduced (1.14) as a coupling condition. This was motivated primarily from continuity considerations; the authors wanted to ensure that a stationary shock infinitesimally close to the junction would remain stationary if perturbed. This is essential for the problem to be *well posed* in the strict sense that the solution should depend continuously on the initial data. The equal pressure condition (1.13) does not have this property [11].

However, one should note that for pipe networks, the junction itself represents a discontinuity in the local topology of the problem; hence the physical relevance of this requirement may be open for debate. In this paper, we will not discuss this issue. Instead, we focus only on the *existence* and *uniqueness* of solutions of the N-junction Riemann problem with constant initial data in each pipe. In this respect, a main result of our current paper is that both the conditions (1.13) and (1.14) fail to provide global existence of solutions if the entropy constraint is taken into account. Furthermore, we propose an alternative coupling condition where unique global existence of entropy solutions is guaranteed.

For the N-junction Riemann problem for (1.8) and (1.9), Colombo and Garavello [11] proved the existence and uniqueness of some stationary solutions and their perturbations when (1.14) is used as coupling condition. The results are shown to be global in space–time and local in the subsonic region of the state space $(\rho, \rho v)$. These results were extended to non-uniform initial data in [12].

Similar local results were achieved by Banda *et al.* [5, 6] for the coupling condition (1.13). Herein, the authors did not consider the entropy constraint through the junction. A unified framework was presented in [13], providing local existence and uniqueness of solutions to the Cauchy problem for general coupling conditions.

In [20], numerical simulations were performed in order to evaluate pressure as momentum-related coupling constant (Eq. (1.13)). The simulations were performed on a tee-shaped junction, as the analytical solution for piecewise constant initial data in this kind of geometry was available from earlier work. As prerequisite for this solution it is stated that for the given geometry and initial data, Eq. (1.7), (1.11) and (1.13) form a well-posed mathematical problem.

Two different flow configurations were considered in the two-dimensional simulations. The first configuration consisted of one ingoing and two outgoing flows, the second of two ingoing and one outgoing flow. The simulation results were averaged and compared to the analytical results. A clear deviance between the simulations and the analytical results was found for the second configuration. Thus, for this configuration the use of geometry and flow-dependent empirical pressure loss coefficients was recommended.

In the present work we propose a momentum-related coupling constant by using the idea of ideal, reversible flow as starting point. Combined with the observation that conservation of mechanical energy is strongly related to conservation of momentum, we suggest to use the Bernoulli invariant, an energy invariant with constant value along streamlines. This allows us to prove global existence both in time and in the subsonic region of state space. Numerical validations of our results have been presented in [27, 26].

1.4. *Outline of the paper*

In Sec. 2, we present the conditions defining the N-junction Riemann problem for the isothermal Euler equations. Further we investigate solutions where the entropy condition is not taken into account. The main result is presented in Theorem 2.13; such solutions exist and are unique whenever the momentum-related coupling constant $\tilde{\mathcal{H}}$ satisfies a monotonicity property. In particular, the constants pressure (1.13) and momentum flux (1.14) have this property.

Section 3 deals with the entropy condition. Results are derived for a three-pipe junction when equal pressure (1.13) and equal momentum flux (1.14) are used as coupling condition. Theorem 3.1 summarizes the findings, that both conditions yield solutions violating the entropy condition in certain ranges of pipe flow rates. Interestingly, there is a perfect duality between these two conditions; for any given velocity distribution, the entropy productions associated with the two different coupling conditions will be of opposite sign.

In Sec. 4, we propose and analyze a new coupling condition; momentum conservation should be replaced with a unique value of the Bernoulli invariant in the junction. Herein, Theorems 4.3 and 4.4 contain our main result; among the three investigated momentum-related coupling constants, only Bernoulli invariant leads to unique existence of entropy solutions for the entire subsonic region of state space.

2. The Riemann Problem at a Junction of N Pipes

We consider a system of N pipes of equal cross-sectional area, connected at a junction as illustrated in Fig. 1. In each segment, the flow is governed by the conservation law (1.1) given by the isothermal Euler equations (1.8) and (1.9). Following [11], we define the N-junction Riemann problem as follows.

Definition 2.1. A solution to the N-junction Riemann problem (1.5) is a set of self-similar functions $\boldsymbol{U}_i(x,t)$ such that

RP0: For all $i \in \{1,\dots,N\}$, there exists a state

$$\boldsymbol{U}_i^*(\bar{\boldsymbol{U}}_1,\dots,\bar{\boldsymbol{U}}_N) = \lim_{x\to 0^+} \boldsymbol{U}_i(x,t) \tag{2.1}$$

such that $\boldsymbol{U}_i(x,t)$ is given by the restriction to $x \in \mathbb{R}^+$ of the Lax solution to the standard Riemann problem for $x \in \mathbb{R}$:

$$\begin{aligned} \frac{\partial \boldsymbol{U}_i}{\partial t} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{U}_i) = 0, \\ \boldsymbol{U}_i(x,0) = \begin{cases} \bar{\boldsymbol{U}}_i & \text{if } x > 0, \\ \boldsymbol{U}_i^* & \text{if } x < 0. \end{cases} \end{aligned} \tag{2.2}$$

RP1: Mass is conserved at the junction:

$$\sum_{i=1}^{N} \rho_i^* v_i^* = 0. \tag{2.3}$$

RP2: There is a unique, scalar momentum-related coupling constant at the junction:

$$\mathcal{H}(\rho_i^*, v_i^*) = \tilde{\mathcal{H}} \quad \forall\, i \in \{1,\dots,N\}. \tag{2.4}$$

Furthermore, *entropy* solutions are defined as follows.

Definition 2.2. An *entropy* solution to the Riemann problem (1.5) is a solution satisfying the conditions RP0–RP2 as well as

RP3: Energy does not increase at the junction, i.e.

$$\sum_{i=1}^{N} \rho_i^* v_i^* \left(\frac{1}{2}(v_i^*)^2 + a^2 \ln \frac{\rho_i^*}{\rho_0} \right) \le 0, \tag{2.5}$$

where ρ_0 is some reference density.

Remark 2.3. The condition RP3 is the isothermal version of the entropy condition proposed in [11]. This will be derived in Sec. 3.1.

2.1. *Uniqueness of solutions*

Given subsonic initial data $\bar{\boldsymbol{U}}_i$, and subsonic states $\boldsymbol{U}_i^*$, the two states in the pipe are connected by a wave of the second family [11]. ρ_i^* is therefore related to v_i^* through an explicit equation [11]. If they are connected by a rarefaction wave, they are related by

$$\ln \frac{\rho_i^*}{\bar{\rho}_i} = M_i^* - \bar{M}_i, \quad \rho_i^* \leq \bar{\rho}_i, \tag{2.6}$$

where we for convenience use the Mach number, $M = v/a$, instead of velocity. Two states connected by a 2-shock curve are related by

$$M_i^* = \bar{M}_i + \left(\sqrt{\frac{\rho_i^*}{\bar{\rho}_i}} - \sqrt{\frac{\bar{\rho}_i}{\rho_i^*}}\right), \quad \rho_i^* > \bar{\rho}_i. \tag{2.7}$$

Using the appropriate relation, we can express the coupling constant $\mathcal{H}(\rho_i^*, M_i^*)$ as a function of one unknown state variable and the initial data. For example, we may use the function $\mathcal{H}_i^*(\rho_i^*; \bar{\rho}_i, \bar{M}_i)$, or written in short form, $\mathcal{H}_i^*(\rho_i^*)$. Before we show results on the uniqueness of solutions, we define a monotonicity property on $\mathcal{H}_i^*$.

Definition 2.4. A coupling constant, $\mathcal{H}_i^*$, is said to be *monotone* if the following conditions are satisfied:

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{R2}} > 0, \quad M_i^* \in \langle -1, 1 \rangle \tag{2.8}$$

and

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{S2}} > 0, \quad M_i^* \in \langle -1, 1 \rangle. \tag{2.9}$$

Herein, the subscript R2 denotes differentiation along the 2-rarefaction curve (2.6) and S2 denotes differentiation along the 2-shock curve (2.7).

The choice of the variable ρ_i^* is here somewhat arbitrary, as demonstrated in the following lemma.

Lemma 2.5. *Monotonicity in ρ_i^* is equivalent to monotonicity in M_i^*. More precisely,*

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{R2}} > 0 \tag{2.10}$$

if and only if

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}M_i^*}\right|_{\mathrm{R2}} > 0. \tag{2.11}$$

Furthermore,

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{S2}} > 0 \tag{2.12}$$

if and only if

$$\left.\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}M_i^*}\right|_{\mathrm{S2}} > 0. \tag{2.13}$$

Proof. The relation between M_i^* and ρ_i^* along a 2-rarefaction curve in Eq. (2.6) may be differentiated to give

$$\frac{\mathrm{d}M_i^*}{\mathrm{d}\rho_i^*} = \frac{1}{\rho_i^*} > 0. \tag{2.14}$$

Similarly, the relation along a 2-shock curve in Eq. (2.7) may be differentiated to give the relation

$$\frac{\mathrm{d}M_i^*}{\mathrm{d}\rho_i^*} = \frac{1}{2\sqrt{\rho_i^*\rho_i}}\left(1 + \frac{\rho_i}{\rho_i^*}\right) > 0. \tag{2.15}$$

The chain rule may then be used to write

$$\frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}M_i^*} = \frac{\mathrm{d}\mathcal{H}_i^*}{\mathrm{d}\rho_i^*}\frac{\mathrm{d}\rho_i^*}{\mathrm{d}M_i^*}. \tag{2.16}$$

□

The following result may then be stated.

Lemma 2.6. *Assume that the state $\bar{\boldsymbol{U}}_i$ and a monotone coupling constant $\mathcal{H}_i^*$ with value $\tilde{\mathcal{H}}$ are given. Then there is a unique state $\boldsymbol{U}_i^*$ with the following properties*:

(1) $\mathcal{H}(\boldsymbol{U}_i^*) = \mathcal{H}_i^*(\rho_i^*) = \tilde{\mathcal{H}}$.
(2) $\boldsymbol{U}_i^*$ *is connected to* $\bar{\boldsymbol{U}}_i$ *with a* 2*-rarefaction curve if* $\mathcal{H}(\bar{\boldsymbol{U}}_i) \geq \tilde{\mathcal{H}}$.
(3) $\boldsymbol{U}_i^*$ *is connected to* $\bar{\boldsymbol{U}}_i$ *with a* 2*-shock curve if* $\mathcal{H}(\bar{\boldsymbol{U}}_i) < \tilde{\mathcal{H}}$.

Proof. The monotone coupling constant in the sense of Theorem 2.4 guarantees that $\mathcal{H}_i^*(\rho_i^*)$ is a monotone function. Hence the uniqueness of $\boldsymbol{U}_i^*$ is proved. The monotonicity also enables the selection of the kind of curve connecting the two states, which is then determined by the Lax-condition. If $\bar{\rho}_i \geq \rho_i^*$ they are connected by a rarefaction wave. Otherwise, if $\bar{\rho}_i < \rho_i^*$ they are connected by a shock wave. □

Remark 2.7. The monotonicity properties assumed in Theorem 2.6 provide the opportunity to express the unknown state variables by inverted functions. If $\bar{\boldsymbol{U}}_i$ and $\boldsymbol{U}_i^*$ are connected by a 2-rarefaction curve, the functions are denoted by the subscript $\mathcal{R}$:

$$\rho_i^* = \rho_{\mathcal{R}}(\mathcal{H}_{i,\mathrm{R2}}^*(\rho_i^*) = \tilde{\mathcal{H}}), \tag{2.17}$$

$$M_i^* = M_{\mathcal{R}}(\mathcal{H}_{i,\mathrm{R2}}^*(\rho_i^*) = \tilde{\mathcal{H}}). \tag{2.18}$$

Similarly, if connected by a 2-shock curve the inverted functions are denoted by the subscript $\mathcal{S}$:

$$\rho_i^* = \rho_{\mathcal{S}}(\mathcal{H}_{i,\mathrm{S2}}^*(\rho_i^*) = \tilde{\mathcal{H}}), \tag{2.19}$$

$$M_i^* = M_{\mathcal{S}}(\mathcal{H}_{i,\mathrm{S2}}^*(\rho_i^*) = \tilde{\mathcal{H}}). \tag{2.20}$$

A stronger result of Theorem 2.6 may be stated when both the initial state, $\bar{\boldsymbol{U}}_i$, and the coupling constant, $\tilde{\mathcal{H}}$ are subsonic.

Proposition 2.8. *Assume that the state* $\bar{\boldsymbol{U}}_i$ *and the coupling constant* $\tilde{\mathcal{H}}$ *are given, where* $\bar{\boldsymbol{U}}_i$ *is subsonic and* $\tilde{\mathcal{H}}$ *satisfies the inequality*

$$\mathcal{H}_i^*|_{\mathrm{R2}}(M_i^* = -1) < \tilde{\mathcal{H}} < \mathcal{H}_i^*|_{\mathrm{S2}}(M_i^* = 1). \tag{2.21}$$

Further, assume that the coupling constant is monotone in the sense of Theorem 2.4. *Then, a state* $\boldsymbol{U}_i^*$ *satisfying* RP0 *is uniquely defined.*

Proof. The results in Theorem 2.6 enable the construction of the functions

$$\rho_i^*(\tilde{\mathcal{H}}) = \begin{cases} \rho_{\mathcal{R}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} < \mathcal{H}(\bar{\boldsymbol{U}}_i), \\ \bar{\rho}_i & \text{if } \tilde{\mathcal{H}} = \mathcal{H}(\bar{\boldsymbol{U}}_i), \\ \rho_{\mathcal{S}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} > \mathcal{H}(\bar{\boldsymbol{U}}_i), \end{cases} \tag{2.22}$$

$$M_i^*(\tilde{\mathcal{H}}) = \begin{cases} M_{\mathcal{R}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} < \mathcal{H}(\bar{\boldsymbol{U}}_i), \\ \bar{M}_i & \text{if } \tilde{\mathcal{H}} = \mathcal{H}(\bar{\boldsymbol{U}}_i), \\ M_{\mathcal{S}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} > \mathcal{H}(\bar{\boldsymbol{U}}_i). \end{cases} \tag{2.23}$$

Note that (2.22) and (2.23) are continuous, monotonically increasing functions, and that the range of M_i^* is $\langle -1, 1\rangle$ in the interval (2.21). Furthermore, the range of ρ_i^* is

$$\rho_i^* \in (\rho_{\mathcal{R}}(\mathcal{H}_i^*(M_i^* = -1)), \rho_{\mathcal{S}}(\mathcal{H}_i^*(M_i^* = 1))). \tag{2.24}$$

Due to the monotonicity property shown in Eqs. (2.14) and (2.15), this range may be expressed by Eqs. (2.6) and (2.7). Note that along a 2-shock curve, Eq. (2.7) may be rearranged to give

$$\rho_i^* = \frac{\bar{\rho}_i}{4}(M_i^* - \bar{M}_i + \sqrt{(M_i^* - \bar{M}_i)^2 + 4})^2. \tag{2.25}$$

Thus Eq. (2.24) can be rewritten as:

$$\rho_i^* \in \left(\bar{\rho}_i \exp(-1 - \bar{M}_i), \frac{\bar{\rho}_i}{4}(1 - \bar{M}_i + \sqrt{(1 - \bar{M}_i)^2 + 4})^2\right). \tag{2.26}$$

The proof is complete. □

The following statement about the solution to the N-junction Riemann problem may then be made.

Proposition 2.9. *Assume that subsonic initial states* $\bar{\boldsymbol{U}}_i$ *are given in each pipe segment* $i \in \{1, \dots, N\}$ *and that the momentum-related coupling constant is monotone in the sense of Definition* 2.4. *Further, assume that a set of solutions,* $\{\boldsymbol{U}_i^*\}$, *exists and that each of the solutions satisfies* RP0–RP2. *Then* $M_i^* \in \langle -1, 1\rangle \forall i$ *if and only if*

$$\max_i \mathcal{H}_i^*|_{\mathrm{R2}}(M_i^* = -1) < \tilde{\mathcal{H}} < \min_i \mathcal{H}_i^*|_{\mathrm{S2}}(M_i^* = 1). \tag{2.27}$$

Proof. First observe that

$$M_i^* \in \langle -1, 1\rangle \quad \forall i \tag{2.28}$$

implies that $\mathcal{H}_i^*(\rho_i^*) = \tilde{\mathcal{H}}$ must lie in the interval (2.21) for all i, thus establishing (2.27). Conversely, if (2.27) holds, it follows from Theorem 2.8 that the solutions $\{\boldsymbol{U}_i^*\}$ are subsonic. □

Note that the assumption that the solution exists is essential here, as (2.27) together with RP0 and RP2 do not necessarily imply RP1.

The *uniqueness* of solutions may now be established.

Proposition 2.10. *Assume that subsonic initial states* $\bar{\boldsymbol{U}}_i$ *are given in each pipe segment* $i \in \{1, \ldots, N\}$ *and that the coupling constant is monotone in the sense of Theorem* 2.4. *If there is a set of subsonic solutions* $\boldsymbol{U}_i^*$ *satisfying* RP0–RP2, *this set is unique.*

Proof. Consider the mass flux as a function of $\tilde{\mathcal{H}}$:

$$(\rho M)_i^*(\tilde{\mathcal{H}}) = \rho_i^*(\tilde{\mathcal{H}}) M_i^*(\tilde{\mathcal{H}}) = \begin{cases} \rho_{\mathcal{R}}(\tilde{\mathcal{H}}) M_{\mathcal{R}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} < \mathcal{H}(\bar{\boldsymbol{U}}_i), \\ \bar{\rho}_i \bar{M}_i & \text{if } \tilde{\mathcal{H}} = \mathcal{H}(\bar{\boldsymbol{U}}_i), \\ \rho_{\mathcal{S}}(\tilde{\mathcal{H}}) M_{\mathcal{S}}(\tilde{\mathcal{H}}) & \text{if } \tilde{\mathcal{H}} > \mathcal{H}(\bar{\boldsymbol{U}}_i). \end{cases} \tag{2.29}$$

Along a 2-rarefaction curve, Eq. (2.14) may be inserted to give

$$\mathrm{d}(\rho M)_i^* = (1 + M_i^*)\, \mathrm{d}\rho_i^*. \tag{2.30}$$

Similarly, along a 2-shock curve, Eq. (2.15) inserted gives

$$\mathrm{d}(\rho M)_i^* = \left(1 + M_i^* + \frac{\left(\sqrt{\rho_i^*} - \sqrt{\rho_i}\right)^2}{2\sqrt{\rho_i^* \rho_i}}\right) \mathrm{d}\rho_i^*. \tag{2.31}$$

It then follows from (2.8) and (2.9) that in the subsonic region, (2.29) is a monotonically increasing function, and in particular the *total* mass flux

$$\mathcal{J}(\tilde{\mathcal{H}}) = \sum_{i=1}^{N} (\rho M)_i^*(\tilde{\mathcal{H}}) \tag{2.32}$$

is a monotonically increasing function of $\tilde{\mathcal{H}}$. This guarantees that there is at most one valid solution to RP1:

$$\mathcal{J}(\tilde{\mathcal{H}}) = 0. \tag{2.33}$$

The proof is complete. □

Although (2.27) is a necessary condition for subsonic solutions to exist, it is not sufficient. We define the *subsonic region* of the initial data as follows.

Definition 2.11. Assume that a set $\{\bar{\boldsymbol{U}}_i\}$ of initial data is given. Assume that this set satisfies the conditions

(1) $\bar{M}_i \in \langle -1, 1\rangle \quad \forall\, i$;
(2) $\mathcal{J}(\mathcal{H}^-) < 0$, where

$$\mathcal{H}^- = \max_i \mathcal{H}_i^*|_{\mathrm{R2}}(M_i^* = -1); \tag{2.34}$$

(3) $\mathcal{J}(\mathcal{H}^+) > 0$, where

$$\mathcal{H}^+ = \min_i \mathcal{H}_i^*|_{\mathrm{S2}}(M_i^* = 1). \tag{2.35}$$

Such a set of initial data is said to belong to the *subsonic region.* Herein we have used the notation of Definition 2.4.

Remark 2.12. Condition (2) and (3) in Theorem 2.11 are important when defining the subsonic region as there exist states that satisfy (2.27) where

$$\mathcal{J}(\mathcal{H}^-) > 0, \tag{2.36}$$

as well as states that satisfy (2.27) where

$$\mathcal{J}(\mathcal{H}^+) < 0. \tag{2.37}$$

Hence Definition 2.11 describes precisely the region where both the initial data and the resulting junction states are subsonic.

The results of this section may be summed up by the following proposition.

Proposition 2.13. *Assume that the initial data* $\bar{\boldsymbol{U}}_i$ *belongs to the subsonic region in the sense of Definition* 2.11 *and that the momentum-related coupling constant is monotone in the sense of Theorem* 2.4. *Then there exists a unique set of subsonic solutions satisfying* RP0–RP2.

Proof. Theorem 2.8 proves the uniqueness of a state $\boldsymbol{U}_i^*$ satisfying RP0 given subsonic initial state, $\bar{\boldsymbol{U}}_i$, and coupling constant, $\tilde{\mathcal{H}}$. Theorem 2.10 proves the uniqueness of the set of solutions $\boldsymbol{U}_i^*$ that satisfies RP0–RP2, given that such a set of solutions exists. Finally, the definition of the subsonic region in Theorem 2.11 guarantees the existence of the unique set of solutions. □

Remark 2.14. The analysis so far has not taken into account the entropy condition, (RP3, Eq. (2.5)). According to Theorem 2.13, a set of initial conditions satisfying RP0–RP2 (Eqs. (2.2), (2.3) and (2.4)) has the unique solution

$$\boldsymbol{U}_i^* = \bar{\boldsymbol{U}}_i. \tag{2.38}$$

If this solution does not satisfy the entropy condition, it is *impossible* to construct an entropy solution to the N-junction Riemann problem defined by the initial condition. The relation between the solution to RP0–RP2 and the entropy condition (RP3) is found in Sec. 3.

2.2. *Monotonicity of specific coupling constants*

Let $\mathcal{H}_{\mathrm{MF}}$ denote momentum flux as momentum-related coupling constant (1.14). For the isothermal Euler equations (1.8) and (1.9) this is equivalent to:

$$\mathcal{H}_{\mathrm{MF}} = \rho(M^2 + 1). \tag{2.39}$$

Similarly, let $\mathcal{H}_p$ denote pressure as coupling constant:

$$\mathcal{H}_{\mathrm{p}} = \rho. \tag{2.40}$$

The following results may then be stated.

Lemma 2.15. *Pressure is a monotone coupling constant in the sense of Theorem* 2.4.

Proof. Inserting the condition of pipe section i into (2.40) we have:

$$\mathcal{H}_{i,p}^*(\rho_i^*) = \rho_i^* \tag{2.41}$$

and accordingly

$$\frac{\mathrm{d}\mathcal{H}_{i,p}^*}{\mathrm{d}\rho_i^*} = 1. \tag{2.42}$$

Thus the coupling constant is monotone. □

Lemma 2.16. *In the subsonic region, momentum flux is a monotone coupling constant in the sense of Theorem* 2.4.

Proof. Along a 2-rarefaction curve, Eq. (2.6) may be inserted into (2.39) to give

$$\mathcal{H}_{i,\mathrm{MF}}^*(\rho_i^*) = \rho_i^* \left(1 + \left(\ln \frac{\rho_i^*}{\bar{\rho}_i} + \bar{M}_i\right)^2\right), \tag{2.43}$$

with corresponding derivative

$$\left.\frac{\mathrm{d}\mathcal{H}_{i,\mathrm{MF}}^*}{\mathrm{d}\rho_i^*}\right|_{\mathrm{R2}} = \left(\left(1 + \ln \frac{\rho_i^*}{\bar{\rho}_i}\right) + \bar{M}_i\right)^2 > 0. \tag{2.44}$$

Along a 2-shock curve, Eq. (2.7) may be inserted to give

$$\mathcal{H}^*_{i,\mathrm{MF}}(\rho^*_i) = \rho^*_i \left(1 + \left(\bar{M}_i + \left(\sqrt{\frac{\rho^*_i}{\bar{\rho}_i}} - \sqrt{\frac{\bar{\rho}_i}{\rho^*_i}}\right)\right)^2\right). \tag{2.45}$$

The derivative is thus

$$\left.\frac{\mathrm{d}\mathcal{H}^*_{i,\mathrm{MF}}}{\mathrm{d}\rho^*_i}\right|_{\mathrm{S2}} = \left(\frac{\rho^*_i}{\bar{\rho}_i} - 1\right)\left(1 + \bar{M}_i\sqrt{\frac{\bar{\rho}_i}{\rho^*_i}}\right) + \left(\bar{M}_i + \sqrt{\frac{\rho^*_i}{\bar{\rho}_i}}\right)^2 + \frac{\rho^*_i}{\bar{\rho}_i} > 0,$$
$$\bar{M}_i, M^*_i \in \langle -1, 1\rangle, \tag{2.46}$$

and consequently the coupling constant is monotone. □

3. Energy Conservation in a Junction

3.1. *The entropy condition*

In the previous section, the monotonicity of the two momentum-related coupling constants pressure (1.13) and momentum flux (1.14) was established to verify the uniqueness of solutions to RP0–RP2. In this section we will investigate if the solutions obtained when using the coupling constants obey the entropy condition (RP3, Eq. (2.5)). The investigation will use the case of a junction with three connected pipes.

The entropy condition originates from the energy flux in the general Euler equations. Due to the isentropic assumption and the pressure law (Eq. (1.10)), the fundamental thermodynamic differential is simplified to

$$\mathrm{d}e = \frac{a^2}{\rho}\mathrm{d}\rho. \tag{3.1}$$

Integrating this equation yields:

$$e = a^2 \ln\left(\frac{\rho}{\rho_0}\right). \tag{3.2}$$

We may then express the energy flux as:

$$v(E + p) = v\rho\left(\frac{1}{2}v^2 + a^2\ln\left(\frac{\rho}{\rho_0}\right) + a^2\right). \tag{3.3}$$

For an N-junction, the total energy flux thus becomes:

$$\begin{aligned} Q &= \sum_{i=1}^{N}\left(v_i\rho_i\left(\frac{1}{2}v_i^2 + a^2\ln\left(\frac{\rho_i}{\rho_0}\right) + a^2\right)\right) \\ &= \sum_{i=1}^{N}\left(v_i\rho_i\left(\frac{1}{2}v_i^2 + a^2\ln(\rho_i)\right)\right), \end{aligned} \tag{3.4}$$

where the terms a^2 and $a^2\ln(\rho_0)$ in (3.4) cancel due to the conservation of mass (2.3).

3.2. *Coupling constant*: *Pressure*

By the assumptions $N = 3$ and pressure as coupling constant, Eqs. (2.40), (2.3) and (3.4) become:

$$\rho_i^* = \tilde{\rho}, \tag{3.5}$$

$$\sum_{i=1}^{3} v_i^* = 0 \tag{3.6}$$

and

$$\begin{aligned} Q &= \tilde{\rho} \sum_{i=1}^{3} \left(v_i^* \left(\frac{1}{2} (v_i^*)^2 + a^2 \ln(\tilde{\rho}) \right) \right) \\ &= \frac{1}{2} \tilde{\rho} \sum_{i=1}^{3} (v_i^*)^3 + a^2 \tilde{\rho} \ln(\tilde{\rho}) \sum_{i=1}^{3} v_i^* \\ &= \frac{1}{2} \tilde{\rho} \sum_{i=1}^{3} (v_i^*)^3, \end{aligned} \tag{3.7}$$

respectively. Equation (3.7) may be expanded to give

$$Q = \frac{1}{2} \tilde{\rho} \left(\left(\sum_{i=1}^{3} v_i^* \right)^3 - 3(v_1^* + v_2^*)(v_2^* + v_3^*)(v_1^* + v_3^*) \right). \tag{3.8}$$

Inserting (3.6) into (3.8) results in the expression

$$Q = \frac{3}{2} \tilde{\rho} v_1^* v_2^* v_3^*. \tag{3.9}$$

Hence, the entropy condition is only fulfilled for one ingoing and two outgoing flows, or for cases with zero flow-rate in one of the pipes.

3.3. *Coupling constant*: *momentum flux*

The assumption of equal momentum flux at the junction, $\rho_i^*(1 + (M_i^*)^2) = \tilde{\mathcal{H}}$, results in the following set of equations:

$$\rho_i^* = \frac{\tilde{\mathcal{H}}}{1 + (M_i^*)^2}, \tag{3.10}$$

$$\sum_{i=1}^{3} \rho_i^* v_i^* = \tilde{\mathcal{H}} a \sum_{i=1}^{3} \frac{M_i^*}{1 + (M_i^*)^2} = 0 \tag{3.11}$$

and

$$\begin{aligned} Q &= \sum_{i=1}^{3} \tilde{\mathcal{H}} a \frac{M_i^*}{1 + (M_i^*)^2} \frac{a^2 ((M_i^*)^2 + 2 \ln(\frac{1}{1 + (M_i^*)^2}))}{2} \\ &= \tilde{\mathcal{H}} a^3 \sum_{i=1}^{3} \frac{M_i^*}{1 + (M_i^*)^2} \frac{((M_i^*)^2 - 2 \ln(1 + (M_i^*)^2))}{2}. \end{aligned} \tag{3.12}$$

As we will see below, the function Q takes the value of zero only when one of the flow velocities is zero. Further, the function is positive for a certain range of flow velocities.

Hence both coupling constants results in unphysical solutions at certain ranges of flow velocities. In addition it should be noted that in the range of flow velocities yielding physical solutions for one condition, the other condition has unphysical solutions.

Proposition 3.1. *In the case of a three-pipe junction, the energy flux function for coupling conditions of equal pressure (Eq. (3.9)) and equal momentum flux (Eq. (3.12)) takes values of opposite sign for all cases with non-zero flow velocities. In particular, for the equal pressure condition, whenever there are two incoming and one outgoing flow the entropy constraint is violated. For the equal momentum flux condition, the entropy constraint is violated whenever there are one incoming and two outgoing flows.*

3.4. *Proof of Proposition* 3.1

We write (3.12) as

$$\hat{Q}(M_k) = \sum_{k=1}^{3} z_k(M_k) b_k(M_k), \tag{3.13}$$

where

$$z_k(M_k) = \frac{M_k}{1 + M_k^2}, \tag{3.14}$$

$$b_k(M_k) = \frac{M_k^2 - 2\ln(1 + M_k^2)}{2}. \tag{3.15}$$

Conservation of mass (3.11) may then be expressed as:

$$\sum_{k=1}^{3} z_k(M_k) = 0. \tag{3.16}$$

The flux function $\hat{Q}$ has two obvious values of z_1 for which it is zero: $z_1 = -z_2$ and $z_1 = 0$. As b_k is a function of M_k^2 only, $b_k(z_k) = b_k(-z_k)$. Thus for $z_1 = -z_2$:

$$z_3 = -(z_1 + z_2) = 0, \tag{3.17}$$

$$b(z_1) = b_1(-z_1) = b_2(z_2), \tag{3.18}$$

$$\begin{aligned}\hat{Q} &= z_1 b_1(z_1) + z_2 b_2(z_2) + z_3 b_3(z_3) \\ &= -z_2 b_2(z_2) + z_2 b_2(z_2) = 0.\end{aligned} \tag{3.19}$$

And for $z_1 = 0$:

$$z_3 = -z_2, \tag{3.20}$$

$$\hat{Q} = z_2 b_2(z_2) - z_2 b_2(z_2) = 0. \tag{3.21}$$

The behavior of $\hat{Q}(z_k)$ may then be found by investigating the derivatives. In the analysis it is assumed that z_2 is a constant, hence only variables related to z_1 and z_3 are included. Now

$$\begin{aligned}\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} &= \frac{\mathrm{d}z_1}{\mathrm{d}z_1}b_1 + z_1\frac{\mathrm{d}b_1}{\mathrm{d}z_1} + \frac{\mathrm{d}z_3}{\mathrm{d}z_1}b_3 + z_3\frac{\mathrm{d}b_3}{\mathrm{d}z_1} \\ &= b_1 + z_1\frac{\mathrm{d}b_1}{\mathrm{d}z_1} - b_3 + z_3\frac{\mathrm{d}b_3}{\mathrm{d}z_3}\frac{\mathrm{d}z_3}{\mathrm{d}z_1} \\ &= (b_1 - b_3) + \left(z_1\frac{\mathrm{d}b_1}{\mathrm{d}z_1} - z_3\frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2\hat{Q}}{\mathrm{d}{z_1}^2} &= \frac{\mathrm{d}}{\mathrm{d}z_1}\left[(b_1 - b_3) + \left(z_1\frac{\mathrm{d}b_1}{\mathrm{d}z_1} - z_3\frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right)\right] \\ &= 2\left(\frac{\mathrm{d}b_1}{\mathrm{d}z_1} + \frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right) + z_1\frac{\mathrm{d}^2b_1}{\mathrm{d}{z_1}^2} + z_3\frac{\mathrm{d}^2b_3}{\mathrm{d}{z_3}^2}. \end{aligned} \tag{3.23}$$

For convenience the derivative $\frac{\mathrm{d}b_k}{\mathrm{d}z_k}$ is found as a function of M_k:

$$\frac{\mathrm{d}z_k}{\mathrm{d}M_k} = \frac{(1+M_k^2) - 2M_k^2}{(1+M_k^2)^2} = \frac{1-M_k^2}{(1+M_k^2)^2}, \tag{3.24}$$

$$\frac{\mathrm{d}M_k}{\mathrm{d}z_k} = \frac{(1+M_k^2)^2}{1-M_k^2}, \tag{3.25}$$

$$\begin{aligned}\frac{\mathrm{d}b_k}{\mathrm{d}z_k} &= \frac{\mathrm{d}b_k}{\mathrm{d}M_k}\frac{\mathrm{d}M_k}{\mathrm{d}z_k} \\ &= \left(M_k - \frac{2M_k}{1+M_k^2}\right)\left(\frac{(1+M_k^2)^2}{1-M_k^2}\right) \\ &= -M_k(1+M_k^2). \end{aligned} \tag{3.26}$$

In the subsonic region, $M \in \langle -1, 1\rangle$ and $z \in \langle -1/2, 1/2\rangle$. The derivative in Eq. (3.22) may be investigated in three different intervals.

3.4.1. *Interval* 1: $z_1 \in \langle -1/2, -z_2]$ *if* $z_2 > 0$

If $z_2 < 0$, $z_1 \in [-z_2, 1/2\rangle$. In both cases $|z_1| \geq |z_2|$ and $|z_1| > |z_3|$ due to Eq. (3.16). The symmetry of b_k as a function of z_k and the sign of its derivative (Eq. (3.26)) gives:

$$b_3(z_3) = b_3(-z_3) > b_1(z_1), \tag{3.27}$$

as well as

$$\left|\frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right| < \left|\frac{\mathrm{d}b_1}{\mathrm{d}z_1}\right|. \tag{3.28}$$

Hence in the first interval

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} = (b_1 - b_3) + \left(z_1 \frac{\mathrm{d}b_1}{\mathrm{d}z_1} - z_3 \frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right) < 0. \tag{3.29}$$

3.4.2. *Interval* 2: $z_1 \in \langle -z_2, 0\rangle$ *if* $z_2 > 0$

If $z_2 < 0$, $z_1 \in \langle 0, -z_2\rangle$. In this interval, Eq. (3.22) is equal to zero for $z_1 = z_3 = -\frac{z_2}{2}$. Possible additional roots are investigated with the aid of Eq. (3.23). The second derivative needed in the last two terms in the equation is found as:

$$\begin{aligned}\frac{\mathrm{d}^2 b_k}{\mathrm{d}{z_k}^2} &= \frac{\mathrm{d}}{\mathrm{d}M_k}[-M(1+M^2)]\frac{\mathrm{d}M_k}{\mathrm{d}z_k} = (-1-3M_k^2)\frac{(1+M_k^2)^2}{1-M_k^2} \\ &= -\frac{(1+3M_k^2)(1+M_k^2)^2}{1-M_k^2} < 0 \quad \text{for } M_k k \in \langle -1, 1\rangle. \end{aligned} \tag{3.30}$$

Equations (3.16), (3.23), (3.26) and (3.30) give the following result: For $z_2 > 0$

$$\frac{\mathrm{d}^2\hat{Q}}{\mathrm{d}{z_1}^2} > 0 \quad \text{for } z_1 \in \langle -z_2, 0\rangle, \tag{3.31}$$

hence $\hat{Q}(z_1 = -z_2/2)$ is the only local minimum for $\hat{Q}$ in the range $z_1 \in \langle -z_2, 0\rangle$ and there are no values of z_1 satisfying the equation $\hat{Q}(z_1) = 0$ in the given interval. For $z_2 < 0$

$$\frac{\mathrm{d}^2\hat{Q}}{\mathrm{d}{z_1}^2} < 0 \quad \text{for } z_1 \in \langle 0, -z_2\rangle, \tag{3.32}$$

hence $\hat{Q}(z_1 = -z_2/2)$ is the only local maximum for $\hat{Q}$ in the range $z_1 \in \langle 0, -z_2\rangle$ and there are no values of z_1 satisfying the equation $\hat{Q}(z_1) = 0$ in the given interval.

3.4.3. *Interval* 3: $z_1 \in [0, 1/2 - z_2\rangle$ *if* $z_2 > 0$

If $z_2 < 0$, $z_1 \in \langle -1/2 - z_2, 0]$. In both cases $|z_1| < |z_3|$ due to Eq. (3.16). The symmetry of b_k as a function of z_k and the sign of its derivative (Eq. (3.26)) gives:

$$b_3(z_3) = b_3(-z_3) < b_1(z_1), \tag{3.33}$$

as well as

$$\left|\frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right| > \left|\frac{\mathrm{d}b_1}{\mathrm{d}z_1}\right|. \tag{3.34}$$

Hence in region three

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} = (b_1 - b_3) + \left(z_1 \frac{\mathrm{d}b_1}{\mathrm{d}z_1} - z_3 \frac{\mathrm{d}b_3}{\mathrm{d}z_3}\right) > 0. \tag{3.35}$$

3.4.4. *Conclusion of proof*

To sum up, it is proved that for $z_2 > 0$:

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} < 0 \quad \text{for } z_1 \in \langle -1/2, -z_2/2 \rangle, \tag{3.36}$$

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} > 0 \quad \text{for } z_1 \in \langle -z_2/2, 1/2 - z_2 \rangle. \tag{3.37}$$

Accordingly, for $z_2 < 0$:

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} > 0 \quad \text{for } z_1 \in \langle -1/2, -z_2/2 \rangle, \tag{3.38}$$

$$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}z_1} < 0 \quad \text{for } z_1 \in \langle -z_2/2, 1/2 - z_2 \rangle. \tag{3.39}$$

Further it is proved that in the interval $z_k \in \langle -1/2, 1/2 \rangle$, $\hat{Q}(z_1) = 0$ only for $z_1 = -z_2$ and $z_1 = 0$.

Assume now that $z_2 > 0$ and $M_2 > 0$. The derivation showed that for a coupling condition of equal momentum flux, the energy flux is non-positive only in the range $z_1 \in \langle -z_2, 0 \rangle$. From Eq. (3.14) it may be deduced that $M_1 = -M_2$ at $z_1 = -z_2$. Hence, from (3.6) and (3.9) it may be found that the energy flux for the coupling condition of equal pressure is non-negative only in the range $z_1 \in \langle -z_2, 0 \rangle$.

For non-positive flows in the second pipe, the argument is similar, but with opposite signs.

4. Proposal for Coupling Condition: Equal Bernoulli Invariant

As pointed out in [14], for a system modeled by the full set of Euler equations, the linear momentum of the fluid may not be conserved at the junction. Hence there is a dependence on the relative position of the pipes. A scalar conserved quantity derived from the vector momentum conservation would therefore be desirable. In classical mechanics, this scalar quantity is the Hamiltonian energy function, and its conservation follows from the underlying symmetries of the equations of motion. The theory is extendable to fluid mechanics [4, 28]; the Euler equations give rise to constants of motion known as *Bernoulli invariants*. For the isothermal Euler equations, the invariant becomes [29, Eq. (3.76)]:

$$B = a^2 \left(\ln \frac{\rho}{\rho_0} + 1 \right) + \frac{1}{2} \boldsymbol{v}^2. \tag{4.1}$$

Since a is a constant, an equivalent invariant is:

$$B = a^2 \ln \frac{\rho}{\rho_0} + \frac{1}{2} \boldsymbol{v}^2. \tag{4.2}$$

4.1. *Existence and uniqueness of solutions when using Bernoulli invariant as coupling constant*

Noticing that ρ_0 is a constant, we may simplify the expression for Bernoulli invariant as coupling constant to:

$$\mathcal{H}_{\mathrm{BI}} = \ln(\rho) + \frac{1}{2}M^2. \tag{4.3}$$

Lemma 4.1. *The Riemann problem at a junction with* RP2 *expressed by Eq.* (4.3) *has a unique solution satisfying* RP0–RP2 *given that the initial data belongs to the subsonic region in the sense of Theorem* 2.11*.*

Proof. To prove the uniqueness of solutions to the Riemann problem at the junction it is sufficient to prove that Bernoulli invariant is a monotone coupling constant in the sense of Theorems 2.4 and 2.5. Existence and uniqueness are then guaranteed by Theorem 2.13.

Along a 2-rarefaction curve, the coupling constant expressed as a function of Mach number is

$$\mathcal{H}^*_{i,\mathrm{BI}}(M^*_i) = M^*_i - \bar{M}_i + \frac{1}{2}(M^*_i)^2 + \ln(\bar{\rho}_i), \tag{4.4}$$

with corresponding derivative:

$$\left.\frac{\mathrm{d}\mathcal{H}^*_{i,\mathrm{BI}}}{\mathrm{d}M^*_i}\right|_{\mathrm{R2}} = 1 + M^*_i > 0 \quad \text{for } M^*_i \in \langle -1, 1\rangle. \tag{4.5}$$

Along a 2-shock curve, the coupling constant is

$$\mathcal{H}^*_{i,\mathrm{BI}}(M^*_i) = \ln\left(\frac{\bar{\rho}_i}{4}\left(M^*_i - \bar{M}_i + \sqrt{\left(M^*_i - \bar{M}_i\right)^2 + 4}\right)^2\right) + \frac{1}{2}\left(M^*_i\right)^2. \tag{4.6}$$

The derivative is

$$\left.\frac{\mathrm{d}\mathcal{H}^*_{i,\mathrm{BI}}}{\mathrm{d}M^*_i}\right|_{\mathrm{R2}} = M^*_i + \frac{2}{\sqrt{\left(M^*_i - \bar{M}_i\right)^2 + 4}}. \tag{4.7}$$

The Lax entropy condition for a 2-shock wave is $\bar{M}_i < M^*_i$. Equation (4.7) may only be negative for negative values of M^*_i and thus only for negative values of $\bar{M}_i$. It is therefore necessary to prove that Eq. (4.7) is positive for all values of $\bar{M}_i \in \langle -1, 0]$, $M^*_i \in \langle -1, 0]$ where $M^*_i - \bar{M}_i > 0$. We apply the notation:

$$f(M^*, \bar{M}) = M^* + \frac{2}{\sqrt{\left(M^* - \bar{M}\right)^2 + 4}}. \tag{4.8}$$

The end-points for f as a function of M^* are:

$$f(M^* = \bar{M}, \bar{M}) = \bar{M} + 1 > 0, \quad \bar{M} \in \langle -1, 0] \tag{4.9}$$

and

$$f(M^* = 0, \bar{M}) = \frac{2}{\sqrt{\bar{M}^2 + 4}} > 0. \tag{4.10}$$

If f is a monotone function of $M^* \in [\bar{M}, 0]$, then the function cannot be negative in this interval. To this end, we find the derivative

$$\frac{\partial f}{\partial M^*} = 1 - \frac{2(M^* - \bar{M})}{((M^* - \bar{M})^2 + 4)^{3/2}}. \tag{4.11}$$

Observing that we now have a function only of $(M^* - \bar{M})$, we replace this by $z \in [0, 1\rangle$. We want to show that

$$1 - \frac{2z}{(z^2 + 4)^{3/2}} > 0, \quad z \in [0, 1\rangle, \tag{4.12}$$

which results in the calculation:

$$\begin{aligned} 1 &> \frac{2z}{(z^2 + 4)^{3/2}}, \\ 2z &< (z^2 + 4)^{3/2}, \\ 4z^2 &< (z^2 + 4)^3. \end{aligned} \tag{4.13}$$

This is easily seen to be true given the possible values of z. □

Unlike the two earlier proposed coupling constants, Bernoulli invariant fulfills the entropy condition in Eq. (3.4).

Proposition 4.2. *When using the Bernoulli invariant as coupling constant, the entropy condition* (*Eq.* (3.4)) *is satisfied for all flow conditions in the general case of N pipes connected at a junction.*

Proof. Inserting $\tilde{\mathcal{H}}$ defined by Eq. (4.3) into the entropy condition and using Eq. (2.3) leads to:

$$\begin{aligned} Q &= a^2 \sum_{i=1}^{N} \rho_i^* v_i^* \left(\frac{1}{2}(M_i^*)^2 + \left(\tilde{\mathcal{H}} - \frac{1}{2}(M_i^*)^2 \right) \right) \\ &= a^2 \tilde{\mathcal{H}} \sum_{i=1}^{N} \rho_i^* v_i^* = 0. \end{aligned} \tag{4.14}$$

□

Finally, the main results may be summed up by the following propositions.

Proposition 4.3. *Consider the N-junction Riemann problem for N pipes with equal cross-sectional areas. With pressure or momentum flux as coupling constant* (RP2) *there exists a unique solution satisfying* RP0–RP2 *provided that the initial data belongs to the subsonic region in the sense of Theorem* 2.11. *There does not exist solutions that satisfy* RP3 (*entropy solutions*) *for all initial data in the subsonic region given by Theorem* 2.11.

Proof. Existence and uniqueness is given by Theorem 2.13 together with Theorems 2.15 and 2.16. Theorem 3.1 shows the lack of entropy solutions for certain intervals of flow rates for $N = 3$. We can extend this negative result to arbitrary N simply by imposing a zero flow velocity in the remaining $N - 3$ pipes. □

Proposition 4.4. *Consider the N-junction Riemann problem for N pipes with equal cross-sectional areas. With Bernoulli invariant as coupling constant* (RP2) *there exists a unique entropy solution satisfying* RP0–RP3 *provided that the initial data belongs to the subsonic region in the sense of Theorem* 2.11*.*

Proof. Existence and uniqueness is given by Theorems 2.13 and 4.1. The result in Theorem 4.2 proves that the solution is entropic. □

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