Calculating the boundary of the estimate of region of attraction

The boundary of the ellipsoid is calculated by solving the equation

$$\frac{1}{2}x^T P x = c \tag{1}$$

This term contain cross terms, in contrast to the original ellipsoid equation

$$\frac{y_1}{a^2} + \frac{y_2}{b^2} = 1$$

where a and b describes the ellipsoid relative to the origin of the y system. This is solved by transforming $x^T P x$ into a diagonal form. Using the facts that P has distinct eigenvalues, there exists a similarity transform

$$M^{-1} P M = \Lambda$$

where M is the eigenvector matrix and Λ is a diagonal matrix of corresponding eigenvalues. Further it is known that M is orthogonal ($MM^T = M^T M = I$) due to P being symmetric. Defining

$$q = Mx \tag{2}$$

the expression $x^T P x$ is rewritten as

$$\begin{aligned} x^T P x &= (Mq)^T P (Mq) \\ &= q^T M^T P M q \\ &= q^T M^{-1} P M q \\ &= q^T \Lambda q \end{aligned} \tag{3}$$

where (2) and the orthogonal property, $MM^T = I \Leftrightarrow M^T = M^{-1}$, has been used. Using (1), (2) and (3) it can be seen that

$$\begin{aligned} \frac{1}{2}q^T \Lambda q &= c \\ &\Leftrightarrow \frac{1}{2c} q^T \Lambda q = 1 \\ &\Leftrightarrow \frac{\lambda_1 q_1}{2c} + \frac{\lambda_2 q_2}{2c} = 1 \\ &\Leftrightarrow \frac{q_1}{\left(\sqrt{\frac{2c}{\lambda_1}}\right)^2} + \frac{q_2}{\left(\sqrt{\frac{2c}{\lambda_2}}\right)^2} = 1 \end{aligned}$$

The ellipsoid is now described for the q system by $a = \sqrt{\frac{2c}{\lambda_1}}$ and $b = \sqrt{\frac{2c}{\lambda_2}}$. It is transformed to the x system through a rotation (since the systems q and x are related to each other through a rotation). Let e_{x_1} and e_{q_1} be unit vectors along x_1 and q_1

$$\begin{aligned} e_{x_1} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ e_{q_1} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

The angle between these vectors/axis (from e_{x_1} to e_{q_1}) is found as

$$\begin{aligned} e_{x_1}^T e_{q_1} &= e_{x_1}^T M e_{x_1} \\ &= m_{11} \\ &= |\overrightarrow{e_{x_1}}|\,|\overrightarrow{e_{q_1}}| \cos\theta \\ &= \cos\theta \end{aligned}$$

which implies that

$$\theta = \cos^{-1} m_{11}$$

is the angle of the q system relative to the x system.