Magnetic Levitation

Bjørnar Bøhagen
Department of Engineering Cybernetics
Norwegian University of Science and Technology
N-7491 Trondheim
NORWAY

2003

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1 Introduction

The magnetic levitation experiment, or for short "maglev", consists of an electromagnet, a ball and a post encased in a rectangular enclosure as shown in Figure 1. One electromagnet pole faces a black post upon which a 2.54 $[cm]$ steel ball

Figure 1: The maglev experiment

rests. The ball elevation from the post is measured using a sensor embedded in the post. The post is designed such that with the ball at rest on its surface, the top of the ball is 12.5 $[mm]$ from the face of the electromagnet (not 14 $[mm]$ as stated in the manual [1]). The purpose of the experiment is to analyse and design controllers that levitates the ball from the post according to a desired set point or a desired trajectory. The post also provide repeatable initial condition for control system performance evaluation. For further description and understanding of the system consult with [1] which is available in the laboratory.

2 Mathematical model

In this section we will derive a mathematical model for the system. First the mechanical dynamics describing the ball is derived, then a model describing the current in the electromagnet is derived. Before deriving the dynamics we define the variables used in this section.

Definition 2.1

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= i \\ \mathbf{x} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \\ u &= E \end{aligned}$$

Be careful not to confuse the state vector, $\mathbf{x}$*, with the position variable,* x*. In the following all vectors will be expressed in bold face.*

2.1 Mechanical model

The forces experienced by the ball are a force due to gravity and a force due to the electromagnet, as illustrated in Figure 2. By using Newton's second law of

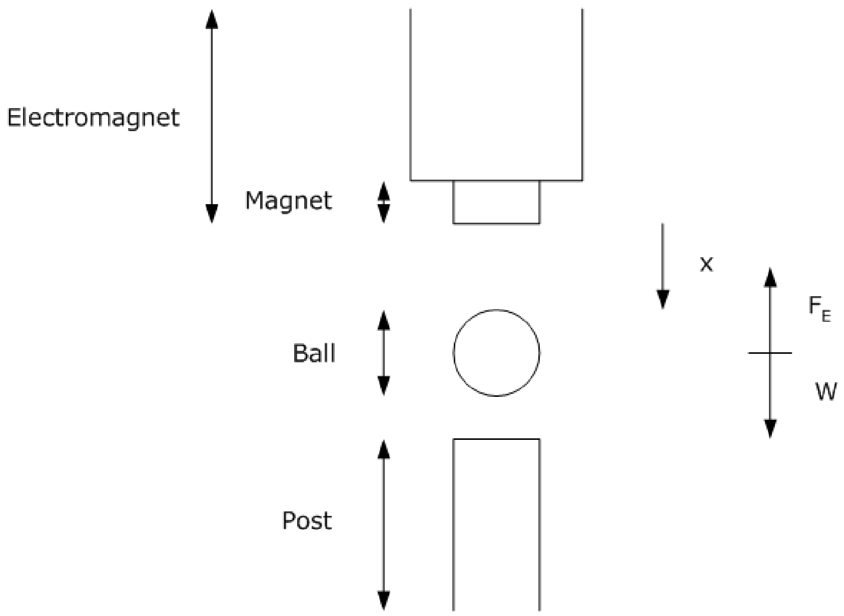

Figure 2: Mechanical system

motion, we may write

$$m\ddot{x} = W - F_E \tag{1}$$

where m is the mass of the ball, x is the position at the top of the ball relative to the electromagnet, W is the weight of the ball and F_E is the force on the ball due to the electromagnet. The forces are given by

$$\begin{aligned} W &= mg & (2) \\ F_E &= K_i \frac{i^2}{(x+d)^2} & (3) \end{aligned}$$

where g is the acceleration due to gravity on the surface of the Earth, i is the current running through the coil in the electromagnet, K_i is the magnetic force constant for the electromagnet/ball pair and d is a constant describing the "point of attack" on the ball due to the electromagnetic force (notice that F_E given here is different from the one given in [1]). By using (2) and (3) we may rewrite (1) as

$$\begin{aligned} m\ddot{x} &= W - F_E \\ &= mg - K_i \frac{i^2}{(x+d)^2} \\ \Leftrightarrow & \quad \ddot{x} = g - \frac{K_i}{m} \frac{i^2}{(x+d)^2} & (4) \end{aligned}$$

2.2 Electrical model

The coil used in the electromagnet is modeled as a inductance and a resistance in series, assuming constant inductance. Further a resistance is coupled in series with the coil (this resistance is used to measure the current through the coil). Figure 3 shows the electrical system. The current loop may be analyzed using Kirchoff's second rule

$$E = R_l i + L\frac{di}{dt} + R_s i$$

where R_l is the resistance in the coil, L is the inductance in the coil and R_s is the series resistance. This equation may be rewritten as

$$\begin{aligned} L\frac{di}{dt} &= E - R_l i - R_s i \\ &= -\left(R_l + R_s\right) i + E \\ \Leftrightarrow & \quad \frac{di}{dt} = -\frac{R_l + R_s}{L} i + \frac{1}{L} E & (5) \end{aligned}$$

2.3 The overall model

When describing and working with the system, its preferable to have formulated it in state space. The state space model will describe both the ball and the current dynamics.

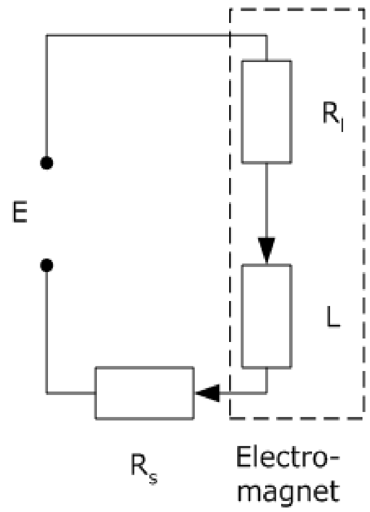

Figure 3: Electrical system

Exercise 1

a) *Derive the state space model* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$.

b) *Derive the equilibrium point* $(\mathbf{x}^*, u^*)$ *and show that*

$$\begin{aligned} x_1^* &= \sqrt{\frac{K_i}{mg}} x_3^* - d & (6)\\ &= \sqrt{\frac{K_i}{mg}} \frac{1}{R_l + R_s} u^* - d & (7)\\ x_2^* &= 0 & (8)\\ x_3^* &= \sqrt{\frac{mg}{K_i}} (x_1^* + d) & (9)\\ &= \frac{1}{R_l + R_s} u^* & (10) \end{aligned}$$

What kind of information can be drawn given a equilibrium point?

2.4 Stability

In the previous section we formulated a state space representation and derived the equilibrium point of the system. We will now analyse the stability of the possible equilibrium points in the open loop system (when u is constant).

Exercise 2

a) *Let* A *denote the Jacobian at the equilibrium point and show that it is* *given by*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & 0 & -a_{23} \\ 0 & 0 & -a_{33} \end{bmatrix} \tag{11}$$

where

$$a_{21} = \frac{2g}{x_1^* + d} \tag{12}$$

$$a_{23} = \sqrt{\frac{gK_i}{m}} \frac{2}{(x_1^* + d)} \tag{13}$$

$$a_{33} = \frac{R_l + R_s}{L} \tag{14}$$

and $a_{ij} \geq 0 \; \forall \{i, j \,| x_1^* \geq 0\}$

b) *Investigate the stability of the open loop system and determine if there are any practical feasible stable equilibrium points (remember that* x_1 *in practice is limited to the set* $x_1 \in [0, 12.5 \cdot 10^{-3}]$*).*

3 Control based on linear methods

In [1] the distributor of the "maglev" experiment suggest a control law for the system based on linear theory. The closed loop system is designed in two steps using pole placement. First a control law for position (levitation) is designed, then a control law for the current loop is designed. When designing the position control loop, the current is used as control input assuming that the closed loop current dynamics (the loop is closed with a control law) is infinite fast relative to the position loop. In this section we will analyse this approach using theory, simulation and finally testing the control law on the actual system. First we define the variables used in this section.

Definition 3.1

$$\begin{aligned}
\bar{u}_1 &= u_1 - u_1^* \\
\bar{x}_1 &= x_1 - x_1^* \\
\bar{x}_2 &= x_2 - x_2^* \\
\bar{x}_3 &= x_3 - x_3^* \\
\bar{x}_{3d} &= x_3 - x_{3d} \\
\bar{x}_4 &= \int \bar{x}_1 dt \\
\bar{x}_5 &= \int \bar{x}_3 dt \\
\bar{x}_6 &= \int\int \bar{x}_1 dt \\
\bar{\mathbf{x}}_1 &= \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_4 \end{bmatrix}^T \\
\bar{\mathbf{x}}_2 &= \begin{bmatrix} \bar{x}_3 & \bar{x}_5 \end{bmatrix}^T \\
\bar{\mathbf{x}} &= \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 & \bar{x}_5 & \bar{x}_6 \end{bmatrix}^T
\end{aligned}$$

where u_1 is the current regarded as control input in the position control loop and u_1^* is the magnitude of the current in equilibrium.

3.1 Position control loop

When considering the position control loop, the current is regarded as input and the current dynamics is ignored. In the following a state feedback setpoint control law will be analyzed (the control loop has a constant reference). Since we are using linear control theory, the model for the position dynamics need to be linearized. The state space model for the position system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g - \frac{K_i}{m}\frac{u_1^2}{(x_1+d)^2} \end{bmatrix}$$

Linearizing this model about a desired equilibrium results in (consult Section 2.4)

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -a_{23} \end{bmatrix} \bar{u}_1$$

In order to achieve integral effect in the closed loop, we extend the state space with the state $\bar{x}_4$. The extended linear system may now be written

$$\begin{aligned} \dot{\bar{\mathbf{x}}}_1 &= \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \bar{\mathbf{x}}_1 + \begin{bmatrix} 0 \\ -a_{23} \\ 0 \end{bmatrix} \bar{u}_1 \\ &= A_1 \bar{\mathbf{x}}_1 + B_1 \bar{u}_1 \end{aligned} \tag{15}$$

A state feedback control law is then given by

$$\begin{aligned} \bar{u}_1 &= -K_1 \bar{\mathbf{x}}_1 \\ &= -k_1 \bar{x}_1 - k_2 \bar{x}_2 - k_4 \bar{x}_4 \\ &= -k_1 \bar{x}_1 - k_2 \dot{\bar{x}}_1 - k_4 \int \bar{x}_1 dt \end{aligned} \tag{16}$$

and it can be seen from (16) that the control law represents a PID for the position.

Remark 3.1
In implementations (simulation and real system) the contribution from the derivative element is reduced at high frequencies. This is to avoid an infinite control input for step changes in the reference and large control input for high frequency disturbances. The derivative element is approximated by

$$\begin{aligned} \frac{k_2 s}{1 + T_d} &= \frac{\frac{k_2}{T_d} s}{\frac{1}{T_d} s + 1} \\ &= \frac{k_2 \omega_{\max} s}{\omega_{\max} + s} \end{aligned}$$

where $\omega_{\max}$ represents the maximum frequency up to which the derivative is active. Below this frequency the approximation acts as a derivative and above this frequency the derivative acts a gain. It can be recognized that $\omega_{\max} = 2\pi f_{\max}$, where the frequency is chosen as $f_{\max} = 200[Hz]$. Since this frequency is high relative to the system dynamics, this approximation will be ignored in our theoretical analyses of the system.

3.2 Current control loop

In the following a state feedback setpoint control law for the current loop will be designed. Since the current dynamics is linear no linearization is required. However, we need to shift the equilibrium to the origin in order to apply our methods for analyzing stability properties of the equilibrium. The current dynamics is given by

$$\begin{aligned}\dot{x}_3 &= -\frac{R_l + R_s}{L} x_3 + \frac{1}{L} u \\ &= -a_{33} x_3 + bu\end{aligned}$$

Using Definition 3.1 the equilibrium is shifted to origin according to

$$\begin{aligned}\dot{\bar{x}}_3 &= \dot{x}_3 - \dot{x}_3^* \\ &= \dot{x}_3 \\ &= -a_{33} x_3 + bu \\ &= -a_{33} \left(\bar{x}_3 + x_3^*\right) + bu \\ &= -a_{33} \bar{x}_3 + bu - a_{33} x_3^* \\ &= -a_{33} \bar{x}_3 + b\left(u - \frac{a_{33}}{b} x_3^*\right) \\ &= -a_{33} \bar{x}_3 + b\left(u - u^*\right) \\ &= -a_{33} \bar{x}_3 + b\bar{u}\end{aligned}$$

where $u^* = \frac{a_{33}}{b} x_3^*$. As in the previous section we desire integral effect in the closed loop. This is achieved by extending the state space with the variable $\bar{x}_5$. The extended linear system may now be written

$$\begin{aligned}\dot{\bar{\mathbf{x}}}_2 &= \begin{bmatrix} -a_{33} & 0 \\ 1 & 0 \end{bmatrix} \bar{\mathbf{x}}_2 + \begin{bmatrix} b \\ 0 \end{bmatrix} \bar{u} \\ &= A_2 \bar{\mathbf{x}}_2 + B_2 \bar{u} \qquad (17)\end{aligned}$$

The state feedback control law is then given by

$$\begin{aligned}\bar{u} &= -K_2 \bar{\mathbf{x}}_2 \\ &= -k_3 \bar{x}_3 - k_5 \bar{x}_5 \\ &= -k_3 \bar{x}_3 - k_5 \int \bar{x}_3 dt \qquad (18)\end{aligned}$$

and it can be seen from (18) that the control law represents a PI for the current.

3.3 The overall control loop

We will now analyse the stability of the proposed control law for the actual nonlinear system. Notice that even though we have designed two control laws, the practical system "sees" only one (the one commanding the input u). The position loop requires a specified current demanded by its control law. This is not in accordance with our previous analysis since the desired current is fed to the current loop as a non constant reference in the actual system, not as a constant reference as assumed in the design of the current control law. This motivates the introduction of a new variable x_{3d} which denotes the desired current from the position loop that is feed as a reference to the current loop and $\bar{x}_{3d} = x_3 - x_{3d}$ which describes the deviation between the actual current the desired current from the position control law.

Exercise 3

a) *Draw a block diagram containing the blocks "Position Control Law", "Current Control Law" and "Plant", and the signals* x_1, x_1^*, $-\bar{x}_1$, $\bar{u}_1$, u_1^*, u_1, x_{3d}, x_3, $-\bar{x}_{3d}$, $\bar{u}$, u^* *and* u. *What kind of "control structure" does* u_1^* *and* u^* *represent.*

b) *Show that*

$$\bar{x}_{3d} = \bar{x}_3 + K_1\bar{\mathbf{x}}_1$$

and that the current control law in the actual system is given by

$$\bar{u} = -k_3K_1\bar{\mathbf{x}}_1 - K_2\bar{\mathbf{x}}_2 - k_1k_5\bar{x}_4 - k_2k_5\bar{x}_1 - k_4k_5\bar{x}_6$$

c) *Show that the actual closed loop system may be written as*

$$\dot{\mathbf{x}} = \begin{bmatrix} \bar{x}_2 \\ g - \frac{K_i\left(\bar{x}_3+x_3^*\right)^2}{m\left(\bar{x}_1+x_1^*+d\right)^2} \\ -a_{33}\bar{x}_3 + b\left(-k_3K_1\bar{\mathbf{x}}_1 - K_2\bar{\mathbf{x}}_2 - k_1k_5\bar{x}_4 - k_2k_5\bar{x}_1 - k_4k_5\bar{x}_6\right) \\ \bar{x}_1 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}$$

d) *Let* $\bar{A}$ *denote the Jacobian calculated at the origin. Calculate* $\bar{A}$.

e) *Get familiar with the m-file initMaglev1.m and calculate the control gains for* $x_1^* = 7[mm]$. *Evaluate the stability of the equilibrium points* $x_1^* = \{3, 6.5, 7, 7.5, 11\}\,[mm]$ *using the control gains found for* $x_1^* = 7[mm]$.

f) *Get familiar with the simulink file simMaglev1.mdl. Simulate the system using a square wave generator with frequency* $0.125[Hz]$ *and amplitudes* 7 ± 4 *and* 7 ± 0.5. *Plot the current versus its reference and the ball position versus its reference.*

g) *Get familiar with the simulink file runMaglev1.mdl and Appendix A and repeat the previous exercise in the laboratory.*

h) *Comment on the result from theory, simulation and laboratory.*

4 Phase plane

In this section we will analyse our system using the phase plane method. Since phase plane analysis apply to second-order systems, the closed loop current dynamics is ignored in the following. Further, we are not able to analyse the PID control law derived in Section 3.1 since this introduced a new state turning the position system into a third order system. However, we will be able to analyse the PD part of the control law. The variables used in this section is given in Definition 4.1.

Definition 4.1

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ \mathbf{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \\ u_1 &= i \end{aligned}$$

Exercise 4

a) *Justify the fact that we are ignoring the closed loop current dynamics.*

b) *Draw a phase portrait of the open loop nonlinear system at equilibrium $x_1^* = 7[mm]$ and comment on the qualitative behavior of the equilibrium.*

c) *Calculate the open loop eigenvalues at the equilibrium $x_1^* = 7[mm]$ and comment.*

d) *Let the equilibrium be given by $x_1^* = 7[mm]$. Draw a phase portrait of the closed loop linearized system using the PD part of the PID control law from Section 3.1. Comment on the qualitative behavior of the equilibrium.*

e) *Calculate the eigenvalues at the equilibrium $x_1^* = 7[mm]$ of the closed loop linearized system and comment.*

5 The describing function method

In this section we will investigate the effect of a backlash-element in the feedback from the position measurement. This implies that the value of the measurement feed to the control law is dependent on wether or not the position is increasing or decreasing. This effect is not critical in this laboratory setup, but appears quite often in mechanical systems. Since our laboratory setup is not bothered with this effect, a backlash-element is inserted in the simulink diagram in the laboratory. We will see that the backlash-element introduces limit cycles in the system. These limit cycles will be analyzed with theory, simulations and testing on the laboratory installation.

The system to be analyzed is given by Figure 7.1 in [2] and it can be recognized that in our case we have $r = 0$, $u = -\psi(\bar{x}_1)$ and $y = \bar{x}_1$. In all of our analysis we will use $x_1^* = 7[mm]$ and let Δ denote the deadband width. Further, we assume that the current control loop in infinite fast relative to the position control loop. This implies that we are only considering position dynamics in theoretical analysis and simulations, taking i as the system input. The position loop is closed with the previously derived state feedback control law (the PID derived in Section 3.1)

Exercise 5

a) *Calculate $G(s) = g_r(s)\, g_p(s)$ where $g_r(s)$ is the control law and $g_p(s)$ is the linearized model. Show, by using Nyquist, that the closed loop system is stable when ignoring the backlash-element.*

b) *Simulate the linearized system and the nonlinear system, when the backlash-element is included, with $\Delta = 0.1 \cdot 10^{-3}$, $\Delta = 0.3 \cdot 10^{-3}$ and $\Delta = 0.5 \cdot 10^{-3}$.*

c) *Conduct tests in the laboratory, when the backlash-element is included, with $\Delta = 0.1 \cdot 10^{-3}$, $\Delta = 0.3 \cdot 10^{-3}$ and $\Delta = 0.5 \cdot 10^{-3}$.*

d) *Use the theory of describing functions to predict whether or not there is a limit cycle in the system, and estimate the amplitude and frequency of possible limit cycles.*

6 Input-output linearization

In this section we will design a control law using input output linearization. In this method the closed loop is designed in two steps. First the control input is used to linearize the system and leaving a fictive input, then the fictive input is used to control the linear system (with for instance pole placement or other linear methods). After having theoretically designed the control law, we will use simulations and laboratory testing to confirm our results. Before designing the control law we define the variables used in this section.

Definition 6.1

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= i \\ u &= E \\ y &= x_1 \\ \mathbf{x} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \end{aligned}$$

6.1 Linearization

The first step in the design consist of linearizing the model using the control input.

Exercise 6

a) *Using Definition 6.1 and the notation from [2] show that*

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \begin{bmatrix} x_2 \\ g - \frac{K_i}{m}\frac{x_3^2}{(x_1+d)^2} \\ -\frac{R_l+R_s}{L}x_3 \end{bmatrix} \\ \mathbf{g}(\mathbf{x}) &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} \\ h(\mathbf{x}) &= x_1 \end{aligned}$$

b) *Show that the system has relative degree $\rho = 3$ in $\{\mathbf{x} \in \mathbb{R}^3 \,|x_3 \neq 0\}$.*

c) *Express $\dddot{y}$ in terms of Lie Derivative and show that*

$$u = -\frac{Lm(x_1+d)^2}{2K_i x_3}v + \frac{Lx_3x_2}{(x_1+d)} + (R_l + R_s)\,x_3$$

in order to satisfy $\dddot{y} = v$.

d) *Explain why we have no zero dynamics. Using the notation from [2] show the following*

$$
\begin{aligned}
A_c &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\
B_c &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
C_c &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

What is this state space representation called?

6.2 Control

This section can be regarded as the control part of the method, a control law for the remaining linear system is designed. We have already created a part of the overall control by using u to linearize the system. What remains is completing the control law by specifying the fictive linear control input v.

Exercise 7

a) *Using the state vector* $\boldsymbol{\zeta} = \begin{bmatrix} \boldsymbol{\xi}^T - \boldsymbol{\xi}^{*T} & \int \zeta_1 dt \end{bmatrix}^T$, *where* $\zeta_1 = y - y^*$ *and* y^* *is a constant reference, show that the state space may be written*

$$
\begin{aligned}
\dot{\zeta} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{\zeta} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} v \\
&= A\boldsymbol{\zeta} + Bv
\end{aligned}
$$

b) *Derive a expression for the state feedback control law* $v = -K\boldsymbol{\zeta}$ *in terms of* y_1 *and* y_1^*. *What kind of control law is this?*

c) *Design* K *using pole placement*

d) *Simulate the system with a square wave generator with frequency* 0.125 *Hz that oscillates around* $x_1^* = 7[mm]$. *How large amplitudes can you achieve?*

e) *Test the control law in the laboratory under the same conditions as above.*

References

[1] Magnetic Levitation Experiment. Quanser Consulting. Advansed Teaching Systems.

[2] Khalil, Hassan K. Nonlinear Systems. Third Edition. Prentice Hall. 2002.