TTK4150 Nonlinear Control Systems
Solution 6
Part 2

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Solution 1

The system is given by

$$\dot{\phi} = R(\phi)\,\omega$$

and

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\,\omega_2\omega_3 + \tau_1 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\,\omega_3\omega_1 + \tau_2 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\,\omega_1\omega_2 + \tau_3 \end{aligned}$$

where $\phi = [\phi_1\ \phi_2\ \phi_3]^T$ *is the orientation vector,* $\omega = [\omega_1\ \omega_2\ \omega_3]^T$ *is a vector of angular velocity along the principal axes,* $\tau = [\tau_1\ \tau_2\ \tau_3]^T$ *is a vector of torque inputs applied about the principal axes and* J_i *are the principal moments of inertia. Further, we have that* $R(\phi) = R^T(\phi)$ *and we assume that* $R(\phi)$ *is non singular in out domain of interest (*$R(\phi)$ *is non singular on* D_ϕ *but not on* $\mathbb{R}^3$*).*

1. *When analyzing a system it is often preferable to express it in compact form of vectors and matrixes to reduce the amount of equations one has to work on. When doing this, working with differential equations on a compact form, it is important to locate properties of the various vectors and matrixes involved in order to take advantage of these properties in the analysis.*

   (a) *The first equation*

   $$\dot{\phi} = R(\phi)\,\omega$$

is already given. Properties connected to this differential equation is that $R(\phi) = R^T(\phi)$ and that $R(\phi)$ is non singular on D_ϕ (which we assume contains our domain of interest). Our task is therefore to find M and $C(\omega)$. It is easily seen that

$$M = diag\{J_1,\ J_2,\ J_3\}$$

and it follows that

$$M = M^T > 0$$

What remains is to show that

$$C(\omega)\omega = \begin{bmatrix} (J_2 - J_3)\omega_2\omega_3 \\ (J_3 - J_1)\omega_3\omega_1 \\ (J_1 - J_2)\omega_1\omega_2 \end{bmatrix}$$

and that $C(\omega) = -C^T(\omega)$. This implies that

$$C(\omega)\omega = \begin{bmatrix} 0 & c_1(\omega) & c_2(\omega) \\ -c_1(\omega) & 0 & c_3(\omega) \\ -c_2(\omega) & -c_3(\omega) & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

and

$$\begin{aligned} c_1(\omega)\omega_2 + c_2(\omega)\omega_3 &= (J_2 - J_3)\omega_2\omega_3 \\ &= J_2\omega_2\omega_3 - J_3\omega_2\omega_3 \\ -c_1(\omega)\omega_1 + c_3(\omega)\omega_3 &= (J_3 - J_1)\omega_3\omega_1 \\ &= J_3\omega_3\omega_1 - J_1\omega_3\omega_1 \\ -c_2(\omega)\omega_1 - c_3(\omega)\omega_2 &= (J_1 - J_2)\omega_1\omega_2 \\ &= J_1\omega_1\omega_2 - J_2\omega_1\omega_2 \end{aligned}$$

where it can be recognized that

$$\begin{aligned} c_1(\omega) &= k_1\omega_3 \\ c_2(\omega) &= k_2\omega_2 \\ c_3(\omega) &= k_3\omega_1 \end{aligned}$$

and

$$\begin{aligned} k_1 + k_2 &= J_2 - J_3 \\ -k_1 + k_3 &= J_3 - J_1 \\ -k_2 - k_3 &= J_1 - J_2 \end{aligned}$$

where it is seen that

$$\begin{aligned} k_1 &= -J_3 \\ k_2 &= J_2 \\ k_3 &= -J_1 \end{aligned}$$

which gives

$$\begin{aligned} C(\omega) &= \begin{bmatrix} 0 & k_1\omega_3 & k_2\omega_2 \\ -k_1\omega_3 & 0 & k_3\omega_1 \\ -k_2\omega_2 & -k_3\omega_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -J_3\omega_3 & J_2\omega_2 \\ J_3\omega_3 & 0 & -J_1\omega_1 \\ -J_2\omega_2 & J_1\omega_1 & 0 \end{bmatrix} \end{aligned}$$

(b) *Since we are solving the stabilization problem, we are working with constant references. The equilibrium of interest is shifted to the origin according to*

$$\begin{aligned} e_1 &= \phi - \phi_d \\ e_2 &= \omega - \omega_d \\ &= \omega \end{aligned}$$

and in order to apply our analyzing tools we will hereafter analyse the differential equations

$$\begin{aligned} \dot{e}_1 &= \dot{\phi} - \dot{\phi}_d \\ &= \dot{\phi} \\ &= R(\phi)\,\omega \\ &= R(\phi)\,e_2 \end{aligned}$$

and

$$\begin{aligned} \dot{e}_2 &= \dot{\omega} \\ &\Rightarrow \; M\dot{e}_2 = C(e_2)\,e_2 + \tau \end{aligned}$$

The stability properties shown for $e = \left[e_1^T \; e_2^T\right]^T = [0\;0]^T$ *will now be the same as the stability properties of the point* $\left[\phi^T \; \omega^T\right]^T = \left[\phi_d^T \; \omega_d^T\right]^T$.

2. *A storage function is given by*

$$V(e) = \frac{1}{2}e_2^T M e_2 + \frac{1}{2}e_1^T K_p e_1$$

where $K_p = K_p^T > 0$.

(a) *The storage function may be interpreted as the total energy of the system, where the term* $\frac{1}{2}e_2^T M e_2$ *describes the kinetic energy of the system (related to motion/velocity) and the term* $\frac{1}{2}e_1^T K_p e_1$ *counts for the potential energy in the system (related to deviation in position/orientation).*

(b) *To find a suitable output based on passivity, we start by calculating the time derivative of the storage function along the solution of the system*

$$\begin{aligned}
\dot{V}(e) &= \frac{1}{2}\dot{e}_2^T M e_2 + \frac{1}{2}e_2^T M \dot{e}_2 + \frac{1}{2}\dot{e}_1^T K_p e_1 + \frac{1}{2}e_1^T K_p \dot{e}_1 \\
&= e_2^T M \dot{e}_2 + e_1^T K_p \dot{e}_1 \\
&= e_2^T \left(C(e_2)e_2 + \tau\right) + e_1^T K_p R(\phi) e_2 \\
&= e_2^T C(e_2) e_2 + e_2^T \tau + e_1^T K_p R(\phi) e_2
\end{aligned}$$

Since $C(e_2)$ *is skew symmetric we have that* $e_2^T C(e_2) e_2 = 0$, *this can be seen by*

$$\begin{aligned}
e_2^T C(e_2) e_2 &= \frac{1}{2}e_2^T C(e_2) e_2 + \frac{1}{2}e_2^T C(e_2) e_2 \\
&= \frac{1}{2}e_2^T C(e_2) e_2 + \frac{1}{2}\left(C(e_2) e_2\right)^T \left(e_2^T\right)^T \\
&= \frac{1}{2}e_2^T C(e_2) e_2 + \frac{1}{2}e_2^T C^T(e_2) e_2 \\
&= \frac{1}{2}e_2^T C(e_2) e_2 - \frac{1}{2}e_2^T C(e_2) e_2 \\
&= 0
\end{aligned}$$

Further, it can be seen that e_2 *is a common factor in the remaining term. Exploiting this leads to the following expression for* $\dot{V}(e)$

$$\begin{aligned}
\dot{V}(e) &= e_2^T \tau + e_1^T K_p R(\phi) e_2 \\
&= e_2^T \tau + \left(e_1^T K_p\right)\left(R(\phi) e_2\right) \\
&= e_2^T \tau + \left(R(\phi) e_2\right)^T \left(e_1^T K_p\right) \\
&= e_2^T \tau + e_2^T R^T(\phi) K_p^T e_1 \\
&= e_2^T \tau + e_2^T R(\phi) K_p e_1 \\
&= e_2^T \left(\tau + R(\phi) K_p e_1\right)
\end{aligned}$$

This last equation suggests that we may choose to make a control law with e_2 as output, which is the angular velocity of the system. This is justified by the fact that we may choose τ to cancel the term $R(\phi) K_p e_1$ and leave a fictive control input, making the system passive from the fictive input to e_2. Note that this choice of control input, τ, does not imply that we are only making use of feedback from the angular velocity, only that we can achieve some passivity properties with respect to ω as output (we are possibly making use of feedback from the orientation to cancel the term $R(\phi) K_p e_1$).

(c) *Since we do not know anything of the sing of $R(\phi)$, it is impossible to conclude any passivity properties of the system with ω as output.*

3. *Using the same storage function as above, we have that*

$$V(e) = \frac{1}{2} e_2^T M e_2 + \frac{1}{2} e_1^T K_p e_1$$

and

$$\dot{V}(e) = e_2^T (\tau + R(\phi) K_p e_1)$$

Since τ is our control input, we are free to choose it as we wish as long as it is defined (not infinity).

(a) *In the case of making the system lossless from v to e_2 we must choose it such that*

$$\dot{V}(e) = e_2^T v$$

This implies that the control input, τ, is taken as

$$\tau = -R(\phi) K_p e_1 + v$$

where v is our new fictive input.

(b) *Since the system is passive with a positive definite storage function*

$$\begin{aligned} V(e) &= \frac{1}{2} e_2^T M e_2 + \frac{1}{2} e_1^T K_p e_1 \\ &\geq \frac{1}{2} \lambda_{\min}(M) \|e_2\|_2^2 + \frac{1}{2} \lambda_{\min}(K_p) \|e_1\|_2^2 \end{aligned}$$

we know that the equilibrium $\left[\phi_d^T \ \omega_d^T\right]^T$ is stable (Lemma 6.6).

(c) *In the case of making the system output strictly passive from* v *to* e_2 *we must choose it such that*

$$\dot{V}(e) \leq e_2^T v - e_2 \rho(e_2), \; e_2 \rho(e_2) > 0 \forall e_2 \neq 0$$

where we desire $e_2 \rho(e_2) = \delta e_2^T e_2$ *in order to draw conclusions with respect to stability. This leads to the choice of*

$$\tau = -R(\phi) K_p e_1 - K_d e_2 + v, \; K_p = K_p^T > 0$$

which results in

$$\begin{aligned} \dot{V}(e) &= e_2^T (\tau + R(\phi) K_p e_1) \\ &= e_2^T (-R(\phi) K_p e_1 - K_d e_2 + v + R(\phi) K_p e_1) \\ &= -e_2^T K_d e_2 + e_2^T v \\ &\leq -\lambda_{\max}(K_d) e_2^T e_2 + e_2^T v \end{aligned}$$

and the system is output strictly passive from v *to* ω *with* $\delta = \lambda_{\max}(K_d)$.

(d) *The control input making the system output strictly passive has added the term* $-K_d e_2$. *This term may be interpreted as adding damping to the system with respect to position since it appears in with respect to the angular velocity* ω. *In the actual system equations it will appear together with the skew symmetric matrix*

$$\begin{aligned} M\dot{e}_2 &= C(e_2) e_2 + \tau \\ &= C(e_2) e_2 - R(\phi) K_p e_1 - K_d e_2 + v \\ &= \underbrace{(C(e_2) - K_d)}_{\text{adding damping}} e_2 - R(\phi) K_p e_1 + v \end{aligned}$$

or

$$M\dot{\omega} = (C(\omega) - K_d)\omega - R(\phi) K_p (\phi - \phi_d) + v$$

(e) *In addition to the stated results due to the property of lossless, we can conclude that the system is finite gain* L_2 *stable from* v *to* ω *with an* L_2 *gain less than or equal to* $1/\delta$ *(Lemma 6.5). To conclude with the stronger result of asymptotically stable, we need to show that the system is zero-state observable. This is confirmed by*

$$\begin{aligned} \omega &\equiv 0 \Leftrightarrow e_2 \equiv 0 \text{ and } \dot{e}_2 \equiv 0 \text{ and } \dot{e}_1 = 0 \\ &\Rightarrow M\dot{e}_2 = (C(e_2) - K_d) e_2 - R(\phi) K_p e_1 = 0 \\ &\Rightarrow -R(\phi) K_p e_1 = 0 \\ &\Rightarrow e_1 = 0 \; \forall \phi \in D_\phi \end{aligned}$$

since $R(\phi)$ is non singular on D_ϕ. Hence, no solution can stay identical in $\omega = 0$ other than the trivial solution $\left[\phi^T(t)\ \ \omega^T(t)\right]^T = \left[\phi_d^T\ 0\right]^T$. This implies that we can conclude asymptotic stability of the equilibrium $(\phi, \omega) = (\phi_d, 0)$ in the system

$$\begin{aligned}\dot{\phi} &= R(\phi)\,\omega \\ M\dot{\omega} &= C(\omega)\,\omega + \tau\end{aligned}$$

where the control input is given by

$$\tau = -R(\phi)\,K_p(\phi - \phi_d) - K_d\omega$$

Even though the storage function is positive definite and radially unbounded, we may not conclude global asymptotic stability of the equilibrium. This is due to the fact that $R(\phi)$ is only guaranteed to be non singular on D_ϕ.

4. *This part is solved by considering properties of feedback systems when both systems possess some passivity properties. Figure 6.11 in Khalil shows the feedback connection applied to draw conclusions. To avoid confusion with respect to notation, the signals e_1 and e_2 in the figure are replaced by z_1 and z_2 respectively.*

   (a) *In order to draw conclusions of asymptotic stability (Lyapunov stability) and robustness (L_2 stability), we at least require the two feedback components to be output strictly passive. This leads to the choice of the control input*

   $$\tau = -R(\phi)\,K_p e_1 - K_d e_2 + v$$

   since this turns the system output strictly passive and zero-state observable with v as input and ω as output. Further, let y_1 and y_2 in the figure represent v and e_2 respectively (system H_2 represents the spacecraft in closed loop with the control law $\tau(\phi, \omega, 0)$ which already is shown asymptotically stable). This implies that u_1 represents measurement noise with respect to the rotational velocity and u_2 represents actuator noise with respect to the control law v.

   (b) *To improve the steady state behavior of the rotational velocity we desire some integral effect with respect to e_2. This can be achieved by using the PID control law*

   $$h_{PID}(s) = K_p\beta\frac{(1 + T_i s)(1 + T_d s)}{(1 + \beta T_i s)(1 + \alpha T_d s)}$$

where $0 \leq T_d \leq T_i$, $1 \leq \beta < \infty$ and $0 < \alpha \leq 1$. By choosing $H_1 = h_{PID}(s)$ we are guaranteed integral effect of the rotational velocity $(-e_2)$. Further, the output of H_1 is given by v which we are free to choose. The proposed PID control law is known to be output strictly passive and zero state observable. Since both dynamic systems are output strictly passive and zero-state observable, we conclude that the equilibrium $(\phi, \omega) = (\phi_d, 0)$ is asymptotically stable. No global results can be concluded since $R(\phi)$ is non singular only on D_ϕ.

It is not possible to use the same approach to cope with a steady state deviation in the orientation. The approach used is based on a relatively strict passivity property with respect to v and ω. Since we do not have any passivity properties with respect to v and $(\phi - \phi_d)$ we are not able to use the same approach.

(c) In order to draw any robustness conclusions with respect to uncertainty in measurements and actuators of the proposed system, we require that both feedback components impose the property of output strictly passive with the output strictly part in the form $\delta y^T y$. Since we do not know if the PID control law imposes this property, we are not able to draw any conclusions.

Solution 2 (Exercise 14.30 a) in Khalil)

The system is given by

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= x_1 + u \end{aligned}$$

where we globally stabilize the origin. If we follow the steps from Khalil, the design of u is done in six steps

- Step one consists of formulating the system to fit a general backstepping form and then use the input to transform the system into an integrator backstepping form. Following the notation of Khalil, it can be recognized that

$$\begin{aligned} \eta &= x_1 \\ \xi &= x_2 \\ f(\eta) &= 0 \\ g(\eta) &= \eta \\ f_a(\eta, \xi) &= \eta \\ g_a(\eta, \xi) &= 1 \end{aligned}$$

and the system is given by

$$\begin{aligned}\dot{\eta} &= f(\eta) + g(\eta)\xi \\ \dot{\xi} &= f_a(\eta,\xi) + g_a(\eta,\xi)u\end{aligned}$$

The system is transformed into a integrator backstepping form by choosing the input as

$$\begin{aligned}u &= \frac{1}{g_a(\eta,\xi)}(u_a - f_a(\eta,\xi)) \\ &= u_a - \eta\end{aligned}$$

where u_a *is considered the input in the integrator backstepping form.*

- *Step two consists of designing a control law* $\phi(\eta)$ *such that the origin of*

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

turns asymptotically stable by using a Lyapunov function. In addition to this we require that $\phi(0) = 0$*. Using a quadratic Lyapunov function, the stability of the origin is and the choice of* $\phi(\eta)$ *is found according to*

$$\begin{aligned}V(\eta) &= \frac{1}{2}\eta^2 \\ \dot{V}(\eta) &= \eta\dot{\eta} \\ &= \eta(f(\eta) + g(\eta)\phi(\eta)) \\ &= \eta^2\phi(\eta) \\ &= -\eta^4\end{aligned}$$

where

$$\phi(\eta) = -\eta^2$$

- *Step three consists of defining a variable* $z = \xi - \phi(\eta)$ *which is used to analyze the fact that* ξ *differs from* $\phi(\eta)$ *in the actual system.*

- *Step four consists of dividing the control input according to*

$$u_a = v + \dot{\phi}$$

- *Step five consists of choosing the control* v *according to*

$$\begin{aligned}v &= -\frac{\partial V}{\partial \eta}g(\eta) - kz \\ &= -\eta\eta - k(\xi - \phi(\eta)) \\ &= -\eta^2 - k\left(\xi + \eta^2\right)\end{aligned}$$

where $k > 0$.

- *Step five consists of summing up the control law according to*

$$\begin{aligned} u &= u_a - \eta \\ &= v + \dot{\phi} - \eta \\ &= -\eta^2 - k\left(\xi + \eta^2\right) - 2\eta\dot{\eta} - \eta \\ &= -\eta^2 - k\left(\xi + \eta^2\right) - 2\eta\eta\xi - \eta \\ &= -\left(1 + k + 2\xi\right)\eta^2 - \eta - k\xi \\ &= -\left(1 + k + 2x_2\right)x_1^2 - x_1 - kx_2 \end{aligned}$$

Since $V(\eta)$ is a radially unbounded Lyapunov function and the results hold globally, we conclude that the system is globally asymptotically stable.

Solution 3 (Exercise 14.31 in Khalil)
The system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 + a + \left(x_1 - a^{1/3}\right)^3 \\ \dot{x}_2 &= x_1 + u \end{aligned}$$

As in the previous exercise the system is in the form of (14.53)-(14.54) with

$$\begin{aligned} f &= a + \left(x_1 - a^{1/3}\right)^3 \\ g &= 1 \\ f_a &= x_1 \\ g_a &= 1 \end{aligned}$$

Take

$$\begin{aligned} \phi(x_1) &= -a - \left(x_1 - a^{1/3}\right)^3 - x_1 \\ V &= \frac{1}{2}x_1^2 \end{aligned}$$

and use (14.56).

Solution 4 (Exercise 14.34 in Khalil)
The system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= x_1 - x_2 - x_1x_3 + u \\ \dot{x}_3 &= x_1 + x_1x_2 - 2x_3 \end{aligned}$$

1. *Since we are evaluating the stability of the origin, let $z_1 = x_1$ and x_2 be considered as input in the first equation*

$$\begin{aligned}\dot{z}_1 &= -x_1 + x_2 \\ &= -z_1 + \phi_1 + z_2\end{aligned}$$

where

$$z_2 = x_2 - \phi_1$$

The system is analyzed according to

$$\begin{aligned}V_1 &= \frac{1}{2}z_1^2 \\ \dot{V}_1 &= z_1\dot{z}_1 \\ &= z_1\left(-z_1 + \phi_1 + z_2\right) \\ &= -z_1^2 + z_1\phi_1 + z_1z_2\end{aligned}$$

by choosing $\phi_1 = 0$ we have that the origin of z_1 is "stable" with

$$\dot{V}_1 = -z_1^2 + z_1z_2$$

The next step consists of including the z_2 dynamics in the analysis

$$\begin{aligned}V_2 &= V_1 + \frac{1}{2}z_2^2 \\ \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 \\ &= -z_1^2 + z_1z_2 + z_2\left(\dot{x}_2 - \dot{\phi}_1\right) \\ &= -z_1^2 + z_1z_2 + z_2\left(x_1 - x_2 - x_1x_3 + u\right) \\ &= -z_1^2 + z_1z_2 + z_2\left(z_1 - z_2 - z_1x_3 + u\right) \\ &= -z_1^2 - z_2^2 + z_2\left(2z_1 - z_1x_3 + u\right)\end{aligned}$$

and by choosing

$$\begin{aligned}u &= -2z_1 + z_1x_3 \\ &= -2x_1 + x_1x_3\end{aligned}$$

it can be recognized that the origin is asymptotically stable. By further investigation we see that the dynamics is globally exponential stable by our choice of u. This is seen from the Lyapunov function for the system

$$\begin{aligned}V &= \frac{1}{2}\left(x_1^2 + x_2^2\right) \\ &= \frac{1}{2}\left\|(x_1, x_2)\right\|_2^2 \\ \dot{V} &= -\left\|(x_1, x_2)\right\|_2^2\end{aligned}$$

2. *We have shown that the two first states in the model will converge exponentially to the origin (even when x_3 is unstable). To show that the overall system is asymptotically stable we must show that the last dynamic equation is asymptotically stale. To do this we apply the theory of cascade system and input-to-state stability, where*

$$\begin{aligned}\dot{x}_3 &= x_1 + x_1 x_2 - 2x_3 \\ &= -2x_3 + z_3\end{aligned}$$

where $z_3 = x_1 + x_1 x_2$. Due to the exponential stability of x_1 and x_2, we have that

$$\begin{aligned}\|z_3\| &= \|x_1 + x_1 x_2\| \\ &\leq \|x_1\| + \|x_1 x_2\| \\ &= \|x_1\| + \|x_1\| \|x_2\| \\ &\leq k_1 \|x_1(0)\| e^{-\gamma_1 (t-t_0)} + k_1 \|x_1(0)\| e^{-\gamma_1 (t-t_0)} k_2 \|x_2(0)\| e^{-\gamma_2 (t-t_0)}\end{aligned}$$

which shows that $\|z_3\|$ is exponentially stable (globally asymptotically stable). Using the theory of cascade systems, we can conclude global asymptotically stability of the overall system if x_3 is input-to-state stable. We start by noticing that the origin of x_3 is globally asymptotically stable when $z_3 = 0$, which implies that the dynamics may be ISS. The ISS property is analyzed with a standard quadratic function

$$\begin{aligned}V_3 &= \frac{1}{2} x_3^2 \\ \dot{V}_3 &= x_3 \dot{x}_3 \\ &= x_3(-2x_3 + z_3) \\ &= -2x_3^2 + x_3 z_3 \\ &= -2(1-\theta) x_3^2 - 2\theta x_3^2 + x_3 z_3 \\ &\leq -2(1-\theta)\|x_3\|_2^2 - 2\theta \|x_3\|_2^2 + \|x_3 z_3\| \\ &= -2(1-\theta)\|x_3\|_2^2 - (2\theta \|x_3\|_2 - \|z_3\|)\|x_3\|_2 \\ &\leq -2(1-\theta)\|x_3\|_2^2, \ \forall 2\theta \|x_3\|_2 \geq \|z_3\|\end{aligned}$$

which shows that the system is input to state stable (z_3 regarded as input) with

$$\begin{aligned}W_3 &= 2(1-\theta)\|x_3\|_2^2 \\ \rho &= \frac{1}{2\theta}\|z_3\| \\ 0 &< \theta < 1\end{aligned}$$