TTK4150 Nonlinear Control Systems
Solution 6
Part 1

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Solution 1 (Exercise 13.1 in Khalil)
The system is given by

$$\begin{aligned} M\ddot{\delta} &= P - D\dot{\delta} + \eta_1 E_q \sin(\delta) \\ \tau \dot{E}_q &= -\eta_2 E_q + \eta_3 \cos(\delta) + E_{FD} \end{aligned}$$

which is rewritten in the form $\dot{x} = f(x) + g(x)u$ *using*

$$\begin{aligned} x_1 &= \delta \\ x_2 &= \dot{\delta} \\ x_3 &= E_q \\ u &= E_{FD} \end{aligned}$$

This results in the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}\left(P - Dx_2 + \eta_1 x_3 \sin(x_1)\right) \\ \dot{x}_3 &= \frac{1}{\tau}\left(-\eta_2 x_3 + \eta_3 \cos(x_1) + u\right) \end{aligned}$$

where it can be seen that

$$\begin{aligned} f(x) &= \begin{bmatrix} x_2 \\ \frac{1}{M}\left(P - Dx_2 + \eta_1 x_3 \sin(x_1)\right) \\ \frac{1}{\tau}\left(-\eta_2 x_3 + \eta_3 \cos(x_1)\right) \end{bmatrix} \\ g(x) &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix} \end{aligned}$$

1. *The output is given by* $y = \delta = x_1 = h(x)$. *The relative degree is found as*

$$\begin{aligned}
y &= x_1 \\
\dot{y} &= \dot{x}_1 \\
&= x_2 \\
\ddot{y} &= \dot{x}_2 \\
&= \frac{1}{M}\left(P - Dx_2 + \eta_1 x_3 \sin(x_1)\right) \\
\dddot{y} &= -\frac{D}{M}\dot{x}_2 + \frac{\eta_1}{M}\dot{x}_3 \sin(x_1) + \frac{\eta_1}{M}x_3 \frac{\partial \sin(x_1)}{\partial x_1}\dot{x}_1 \\
&= -\frac{D}{M}\frac{1}{M}\left(P - Dx_2 + \eta_1 x_3 \sin(x_1)\right)\sin(x_1) \\
&\quad + \frac{\eta_1}{\tau M}\sin(x_1)\left(-\eta_2 x_3 + \eta_3 \cos(x_1) + u\right) \\
&\quad + \frac{\eta_1}{M}x_3 \cos(x_1)\, x_2
\end{aligned}$$

The region D_0 *on which the system has relative degree* $\rho = 3$ *is found from*

$$\begin{aligned}
L_g L_f^{\rho-1} h(x) &= L_g L_f^2 h(x) \\
&= \frac{\eta_1}{\tau M}\sin(x_1) \\
&\neq 0 \ \forall x \in D_0
\end{aligned}$$

where $D_0 = \{x \in \mathbb{R}^3 | \sin(x_1) \neq 0\}$. *External variables of the normal form is given by evaluating the Lie Derivative of* h *with respect to* f

$$\begin{aligned}
\xi_1 &= h(x) \\
&= x_1 \\
\xi_2 &= L_f h(x) \\
&= x_2 \\
\xi_3 &= L_f^2 h(x) \\
&= \frac{1}{M}\left(P - Dx_2 + \eta_1 x_3 \sin(x_1)\right)
\end{aligned}$$

Since the relative degree equals the dimension of the system, we have no internal dynamics and the system is minimum phase.

2. *The output is given by* $y = \delta + \gamma\dot{\delta} = x_1 + \gamma x_2 = h(x)$ *where* $\gamma \neq 0$.

The relative degree is found as

$$
\begin{aligned}
y &= x_1 + \gamma x_2 \\
\dot{y} &= \dot{x}_1 + \gamma \dot{x}_2 \\
&= x_2 + \gamma \frac{1}{M}\left(P - Dx_2 + \eta_1 x_3 \sin\left(x_1\right)\right) \\
&= \left(1 - \frac{\gamma D}{M}\right) x_2 + \frac{\gamma \eta_1}{M} x_3 \sin\left(x_1\right) + \gamma P \frac{1}{M} \\
\ddot{y} &= \frac{\partial \dot{y}}{\partial x} \dot{x} \\
&= \left[\begin{array}{ccc} \frac{\gamma\eta_1}{M} x_3 \cos\left(x_1\right) & \left(1 - \frac{\gamma D}{M}\right) & \frac{\gamma \eta_1}{M} \sin\left(x_1\right) \end{array}\right] \dot{x} \\
&= \frac{\gamma \eta_1}{M} x_3 \cos\left(x_1\right) x_2 \\
&\quad + \left(1 - \frac{\gamma D}{M}\right) \frac{1}{M}\left(P - Dx_2 + \eta_1 x_3 \sin\left(x_1\right)\right) \\
&\quad + \frac{\gamma \eta_1}{\tau M} \sin\left(x_1\right)\left(-\eta_2 x_3 + \eta_3 \cos\left(x_1\right) + u\right)
\end{aligned}
$$

The region D_0 on which the system has relative degree $\rho = 2$ is found from

$$
\begin{aligned}
L_g L_f^{\rho-1} h\left(x\right) &= L_g L_f^{1} h\left(x\right) \\
&= \frac{\gamma \eta_1}{\tau M} \sin\left(x_1\right) \\
&\neq 0 \ \forall x \in D_0
\end{aligned}
$$

where $D_0 = \left\{x \in \mathbb{R}^3 \mid \sin\left(x_1\right) \neq 0\right\}$. External variables of the normal form is found by evaluating the Lie Derivative of h with respect to f

$$
\begin{aligned}
\xi_1 &= h\left(x\right) \\
&= x_1 + \gamma x_2 \\
\xi_2 &= L_f h\left(x\right) \\
&= \frac{\partial h\left(x\right)}{\partial x} f\left(x\right) \\
&= \left[\begin{array}{ccc} 1 & \gamma & 0 \end{array}\right] f\left(x\right) \\
&= x_2 + \frac{\gamma}{M}\left(P - Dx_2 + \eta_1 x_3 \sin\left(x_1\right)\right)
\end{aligned}
$$

The internal dynamics $\eta = \phi\left(x\right)$ is chosen to satisfy $\frac{\partial \phi(x)}{\partial x} g\left(x\right) = 0$ and the existence of $T^{-1}\left(x\right)$ in D_0. It can be verified that $\phi\left(x\right) = x_1$ meets *these conditions. With $\phi(x) = x_1$ we have that*

$$\begin{aligned} \dot{\eta} &= \dot{\phi}(x) \\ &= \dot{x}_1 \\ &= x_2 \\ &= \frac{1}{\gamma}(\xi_1 - \eta) \\ &= f_0(\eta, \xi) \end{aligned}$$

The system is said to be minimum phase if the zero dynamics, $\dot{\eta} = f_0(\eta, 0)$, has an asymptotically stable equilibrium point in the domain of interest. From $\dot{\eta} = f_0(\eta, 0) = -\frac{1}{\gamma}\eta$ it can be recognized that the origin of η is asymptotically stable.

Solution 2 (Exercise 13.2 in Khalil)
The system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - x_3 \\ \dot{x}_2 &= -x_1 x_3 - x_2 + u \\ \dot{x}_3 &= -x_1 + u \\ y &= x_3 \end{aligned}$$

Rewriting this model on the form $\dot{x} = f(x) + g(x)u$ results in

$$\begin{aligned} f(x) &= \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix} \\ g(x) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

1. *The relative degree is found from*

$$\begin{aligned} y &= x_3 \\ \dot{y} &= \dot{x}_3 \\ &= -x_1 + u \end{aligned}$$

which shows that the system has relative degree 1 in $\mathbb{R}^3$. Hence, the system is input-output linearizable.

2. *The external part of the normal form is given by*

$$\begin{aligned}\xi_1 &= h(x)\\ &= x_3\end{aligned}$$

To find the internal dynamics we start by setting up the requirements on $\frac{\partial \phi_i}{\partial x}$

$$\begin{aligned}\frac{\partial \phi_1}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_1}{\partial x_3} \end{bmatrix} g(x)\\ &= \frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_1}{\partial x_3}\\ &= 0\\ \frac{\partial \phi_2}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x_3} \end{bmatrix} g(x)\\ &= \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_2}{\partial x_3}\\ &= 0\end{aligned}$$

By choosing

$$\begin{aligned}\phi_1(x) &= x_1\\ \phi_2(x) &= x_2 - x_3\end{aligned}$$

we obtain a global diffeomorphism

$$\begin{aligned}T(x) &= \begin{bmatrix} x_1 \\ x_2 - x_3 \\ x_3 \end{bmatrix}\\ &= \begin{bmatrix} 1 & & \\ & 1 & -1 \\ & & 1 \end{bmatrix} x\end{aligned}$$

and a normal form

$$\begin{aligned}\dot{\eta}_1 &= \dot{x}_1\\ &= -\eta_1 + \eta_2\\ \dot{\eta}_2 &= \dot{x}_2 - \dot{x}_3\\ &= -x_1 x_3 - x_2 + u + x_1 - u\\ &= -\eta_1 \xi_1 - (\eta_2 + x_3) + \eta_1\\ &= \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1\\ \dot{\xi}_1 &= \eta_1 + u\end{aligned}$$

3. *To investigate if the system is minimum phase, we analyze the zero dynamics*

$$\begin{aligned}\dot{\eta} &= f_0(\eta,\xi)|_{\xi=0} \\ &= \left.\begin{bmatrix} -\eta_1+\eta_2 \\ \eta_1-\eta_2-\xi_1-\eta_1\xi_1 \end{bmatrix}\right|_{\xi=0} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\eta\end{aligned}$$

where it can be seen that the origin is not asymptotically stable. Hence, the system is not minimum phase.

Solution 3

The system is rewritten as

$$\begin{aligned}\dot{x} &= f(x)+g(x)u \\ y &= h(x)\end{aligned}$$

where

$$\begin{aligned}f(x) &= \begin{bmatrix} -x_1 \\ x_1x_2 \\ x_2 \end{bmatrix} \\ g(x) &= \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} \\ h(x) &= x_3\end{aligned}$$

1. *The relative degree is found by derivative y with respect to time*

$$\begin{aligned}y &= x_3 \\ \dot{y} &= \dot{x}_3 \\ &= x_2 \\ \ddot{y} &= \dot{x}_2 \\ &= x_1x_2+u\end{aligned}$$

where it can be seen that the system has a relative degree $\rho = 2$ in $x \in \mathbb{R}^2$. Since $\ddot{y} = L_f^2h(x) + L_gL_fh(x)u$ we have that

$$\begin{aligned}L_f^{\rho}h(x) &= L_f^2h(x) \\ &= x_1x_2 \\ L_gL_f^{\rho-1}h(x) &= L_gL_fh(x) \\ &= 1\end{aligned}$$

2. *Since the system has a relative degree it is input-output linearizable.*

3. *The variables for the external dynamics are found according to*

$$\begin{aligned}\xi_1 &= h(x)\\ &= x_3\\ \xi_2 &= L_f h(x)\\ &= \frac{\partial h(x)}{\partial x} f\\ &= x_2\end{aligned}$$

The coordinates for the internal dynamics is chosen such that $T(x)$ *is diffeomorphism on* $\mathbb{R}^3$ *and* $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ *on* $\mathbb{R}^3$*, where* $[\eta, \xi^T]^T = [\phi(x), \psi(x)] = T(x)$*. In addition to this we require* $\phi(0) = 0$ *in order to have the origin as equilibrium. We start by calculating*

$$\begin{aligned}\frac{\partial \phi(x)}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi(x)}{\partial x_1} & \frac{\partial \phi(x)}{\partial x_2} & \frac{\partial \phi(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix}\\ &= \frac{\partial \phi(x)}{\partial x_1} e^{x_2} + \frac{\partial \phi(x)}{\partial x_2}\\ &= 0\end{aligned}$$

and based on these calculations we try

$$\begin{aligned}\frac{\partial \phi(x)}{\partial x_1} &= 1\\ \frac{\partial \phi(x)}{\partial x_2} &= -e^{x_2}\end{aligned}$$

which implies that

$$\phi(x) = x_1 - e^{x_2} + c$$

where c *is some constant. This constant is chosen to satisfy our requirement* $\phi(0) = 0$

$$\begin{aligned}\phi(0) &= -e^0 + c\\ &= -1 + c\\ &\Rightarrow c = 1\end{aligned}$$

Our resulting coordinate transformation is now given by

$$\begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x_1 - e^{x_2} + 1 \\ x_3 \\ x_2 \end{bmatrix}$$

and it can be recognized that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} \eta + e^{\xi_2} - 1 \\ \xi_2 \\ \xi_1 \end{bmatrix}$$

and consequently the inverse transformation exists. Further, it can be recognized that $T(x)$ and $T^{-1}(x)$ are continuously differentiable. Hence, $T(x)$ is diffeomorphism on $\mathbb{R}^3$ and $T(0) = T^{-1}(0) = 0$.

4. *The system may be rewritten as*

$$\begin{aligned} \dot{\eta} &= \dot{x}_1 - \frac{\partial e^{x_2}}{\partial x_2}\dot{x}_2 \\ &= -x_1 + e^{x_2}u - e^{x_2}(x_1 x_2 + u) \\ &= -x_1 - e^{x_2}x_1 x_2 \\ &= -(\eta + e^{x_2} - 1) - e^{x_2}(\eta + e^{x_2} - 1)x_2 \\ &= \left(1 - \eta - e^{\xi_2}\right) + \left(1 - \eta - e^{\xi_2}\right)e^{\xi_2}\xi_2 \\ &= \left(1 - \eta - e^{\xi_2}\right)\left(1 + e^{\xi_2}\xi_2\right) \end{aligned}$$

and

$$\begin{aligned} \dot{\xi} &= A_c\xi + B_c\gamma(x)(u - \alpha(x)) \\ y &= C_c\xi \end{aligned}$$

where

$$\begin{aligned} A_c &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ B_c &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C_c &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \gamma(x) &= L_g L_f h(x) \\ &= 1 \\ \alpha(x) &= -\frac{L_f^2 h(x)}{L_g L_f h(x)} \\ &= -\frac{x_1 x_2}{1} \\ &= -x_1 x_2 \end{aligned}$$

5. *The zero dynamics is given by*

$$\begin{aligned}\dot{\eta} &= f_0(\eta,\xi)\big|_{\xi=0} \\ &= \left(1-\eta-e^{\xi_2}\right)\left(1+e^{\xi_2}\xi_2\right)\Big|_{\xi=0} \\ &= (1-\eta-1)(1+0) \\ &= -\eta\end{aligned}$$

which has an asymptotically stable equilibrium at the origin.

6. *The external dynamics is given by*

$$\dot{\xi} = A_c\xi + B_c\gamma(x)(u-\alpha(x))$$

By choosing

$$u = \gamma^{-1}(x)v + \alpha(x)$$

the zero dynamics is given by

$$\dot{\xi} = A_c\xi + B_c v$$

Since the system is controllable, $rank([A,\ AB]) = 2$*, it can be stabilized by a control input* $v = -K\xi$ *where* K *is chosen such that* $(A_c - B_cK)$ *is Hurwitz. Since* $\dot{\eta} = f_0(\eta,\xi)|_{\xi=0}$ *is asymptotically stable, the origin of the entire system is asymptotically stable.*

7. *Let*

$$\begin{aligned}R &= \begin{bmatrix} r \\ \dot{r}\end{bmatrix} \\ e &= \xi - R \\ u &= \gamma^{-1}(x)(v+\dot{r}) + \alpha(x)\end{aligned}$$

The system is rewritten as

$$\begin{aligned}\dot{\eta} &= f_0(\eta, e+R) \\ \dot{e} &= A_c e + B_c v\end{aligned}$$

and since (A_c, B_c) *is controllable, the loop is closed with* $v = -K\xi$ *where* K *is chosen such that* $(A_c - B_cK)$ *is Hurwitz. Since* $\dot{\eta} = f_0(\eta,\xi)|_{\xi=0}$ *is asymptotically stable, the origin of the entire system is asymptotically stable.*