TTK4150 Nonlinear Control Systems
Solution 5
Part 2

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Solution 1

1. *The nonlinear element is given by*

$$\psi(y) = y^5$$

From Figure 1 it can be recognized that the nonlinearity is a time-

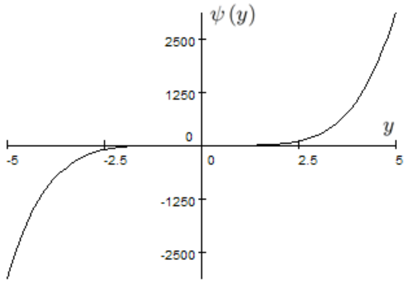

Figure 1: $\psi(y) = y^5$

invariant, memoryless and odd function. The describing function is *calculated as*

$$
\begin{aligned}
\Psi(a) &= \frac{2}{a\pi}\int_0^{\pi}\psi(a\sin(\theta))\sin(\theta)\,d\theta \\
&= \frac{2}{a\pi}\int_0^{\pi}(a\sin(\theta))^5\sin(\theta)\,d\theta \\
&= \frac{2}{a\pi}\int_0^{\pi}a^5\sin^6(\theta)\,d\theta \\
&= \frac{2a^4}{\pi}\int_0^{\pi}\sin^6(\theta)\,d\theta \\
&= \frac{2a^4}{\pi}\frac{5}{16}\pi \\
&= \frac{5a^4}{8}
\end{aligned}
$$

2. *The nonlinear element is given by*

$$\psi(y) = y^3|y|$$

From Figure 2 it can be recognized that the nonlinearity is a time-

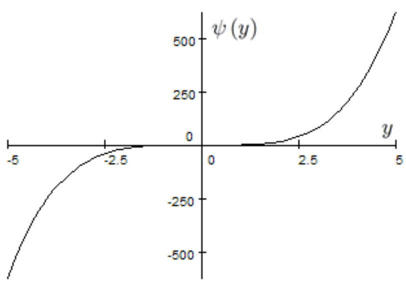

Figure 2: $\psi(y) = y^3|y|$

invariant, memoryless and odd function. The describing function is *calculated as*

$$\begin{aligned}
\Psi(a) &= \frac{2}{a\pi}\int_0^{\pi} \psi(a\sin(\theta))\sin(\theta)\,d\theta \\
&= \frac{2}{a\pi}\int_0^{\pi} (a\sin(\theta))^3 |a\sin(\theta)| \sin(\theta)\,d\theta \\
&= \frac{2}{a\pi}\int_0^{\pi} a^3\sin^3(\theta)\,|a|\,|\sin(\theta)|\sin(\theta)\,d\theta \\
&= \frac{2a^3}{\pi}\int_0^{\pi} |\sin(\theta)|\sin^4(\theta)\,d\theta \\
&= \frac{2a^3}{\pi}\int_0^{\pi} \sin^5(\theta)\,d\theta \\
&= \frac{2a^3}{\pi}\frac{16}{15} \\
&= \frac{32a^3}{15\pi}
\end{aligned}$$

3. *The nonlinear element can be expressed as*

$$\psi(y) = ky + Asgn(y)$$

and the describing function is found as

$$\begin{aligned}
\Psi(a) &= \frac{2}{a\pi}\int_0^{\pi} \psi(a\sin(\theta))\sin(\theta)\,d\theta \\
&= \frac{2}{a\pi}\int_0^{\pi} (ka\sin(\theta) + Asgn(a\sin(\theta)))\sin(\theta)\,d\theta \\
&= \frac{2}{a\pi}\int_0^{\pi} ka\sin(\theta)\sin(\theta)\,d\theta + \frac{2}{a\pi}\int_0^{\pi} Asgn(a\sin(\theta))\sin(\theta)\,d\theta \\
&= \frac{2k}{\pi}\int_0^{\pi} \sin^2(\theta)\,d\theta + \frac{2A}{a\pi}\int_0^{\pi} \sin(\theta)\,d\theta \\
&= \frac{2k}{\pi}\frac{1}{2}\pi + \frac{2A}{a\pi}2 \\
&= k + \frac{4A}{a\pi}
\end{aligned}$$

4. *When* $a \leq \Delta$ *we have that* $\psi(y) = 0$ *and consequently* $\Psi(a) = 0$. *When* $a > \Delta$ *we have that*

$$\psi(y) = \begin{cases} 0 & \text{when } 0 \leq \theta \leq \alpha \text{ and } \pi - \alpha \leq \theta \leq \pi \\ A & \text{when } \alpha < \theta < \pi - \alpha \end{cases}$$

where

$$a \sin(\alpha) = \Delta$$

The describing function is found as

$$\begin{aligned}
\Psi(a) &= \frac{2}{a\pi} \int_0^{\pi} \psi(a \sin(\theta)) \sin(\theta)\, d\theta \\
&= \frac{2}{a\pi} \int_{\alpha}^{\pi/2} \psi(a \sin(\theta)) \sin(\theta)\, d\theta \\
&= \frac{2}{a\pi} \int_{\alpha}^{\pi/2} A \sin(\theta)\, d\theta \\
&= \frac{2A}{a\pi} \cos\alpha \\
&= \frac{2A}{a\pi} \sqrt{1 - \frac{\Delta^2}{a^2}}
\end{aligned}$$

Solution 2

The describing function of the nonlinearity $\psi(\cdot)$ is given by

$$\Psi(a) = \frac{5a^4}{8}$$

1. *Figure 3 shows a Bode diagram of the transfer function $h(s)$. As can*

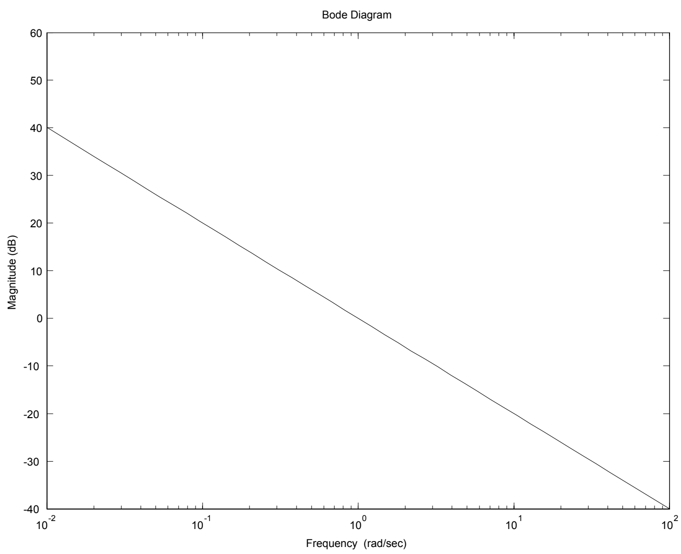

Figure 3: Bode diagram of $h(s)$

be seen from the figure, the system has a low pass characteristic ($h(s)$ is strictly proper). This justifies the use of the describing function method (the simplification of ignoring higher order Fourier coefficients is valid since the effect of these will be reduced due to the low pass of the plant).

2. *If the harmonic balance equation*

$$h(j\omega)\Psi(a) + 1 = 0$$

has a solution, then the closed loop system has a periodic solution (or at least this is a strong implication of a periodic solution). Using that $\Psi(a)$ is real, the complex harmonic balance equation is divided in to two real equations

$$\begin{aligned} \mathrm{Re}[h(j\omega)]\Psi(a) + 1 &= 0 \\ \mathrm{Im}[h(j\omega)] &= 0 \end{aligned}$$

where the second equation is solved first to determine possible frequencies of oscillations. For each solution of ω, if any, the first equation is used to determine the amplitude of the periodic solution a. In our case we have that

$$\begin{aligned} h(j\omega) &= \frac{1 - j\omega}{j\omega(j\omega + 1)} \\ &= \frac{j(1 - j\omega)(-j\omega + 1)}{-\omega(\omega^2 + 1)} \\ &= \frac{j(1 - j2\omega - \omega^2)}{-\omega(\omega^2 + 1)} \\ &= \frac{2\omega + j - j\omega^2}{-\omega(\omega^2 + 1)} \\ &= \frac{-2}{(\omega^2 + 1)} + j\frac{\omega^2 - 1}{\omega(\omega^2 + 1)} \end{aligned}$$

Solving the harmonic balance equation with respect to ω results in

$$\begin{aligned} \mathrm{Im}[h(j\omega)] &= \frac{\omega^2 - 1}{\omega(\omega^2 + 1)} = 0 \\ &\Leftrightarrow \omega^2 - 1 = 0 \\ &\Rightarrow \omega = \pm 1 \end{aligned}$$

and solving the harmonic balance equation with respect to a results in

$$\begin{aligned}
\operatorname{Re}\left[h\left(j\omega\right)\right]\Psi\left(a\right)+1\big|_{\omega=1} &= \left.\frac{-2}{\left(\omega^2+1\right)}\right|_{\omega=1}\frac{5a^4}{8}+1 \\
&= \frac{-2}{2}\frac{5a^4}{8}+1 \\
&= -\frac{5a^4}{8}+1 \\
&= 0 \\
\Rightarrow \quad a &= \sqrt[4]{\frac{8}{5}} = 1.1247
\end{aligned}$$

Hence we have strong implications that a periodic solution exist in this system, and an estimate of the frequency and amplitude is given by

$$\begin{aligned}
\omega &= 1 \\
a &= 1.1247
\end{aligned}$$

3. *From the harmonic balance equation it can be seen that a solution exists if*

$$h\left(j\omega\right) = -\frac{1}{\Psi\left(a\right)}$$

for some ω and a. This is usually investigated in a Nichols diagram where $h\left(j\omega\right)$ is plotted as a function of ω and $-\frac{1}{\Psi(a)}$ is plotted as a function of a. Such diagrams are shown in Figure 4 and Figure 5 where it can be seen that the point of intersection is consistent with the result found when using an analytic approach.

4. *From the figures it can be seen that the periodic solution is stable.*

Solution 3

1. *Using the notation in Khalil, we have that*

$$\begin{aligned}
G\left(s\right) &= h_r\left(s\right)h_p\left(s\right) \\
&= \frac{K}{s}\frac{1}{1+Ts} \\
&= \frac{K}{s\left(1+Ts\right)}
\end{aligned}$$

where $K>0$ and $T>0$. It can be seen that $G\left(s\right)$ has law pass characteristic, by which we conclude that the describing function method can be applied. A bode diagram of $G\left(s\right)$ is shown in Figure 6

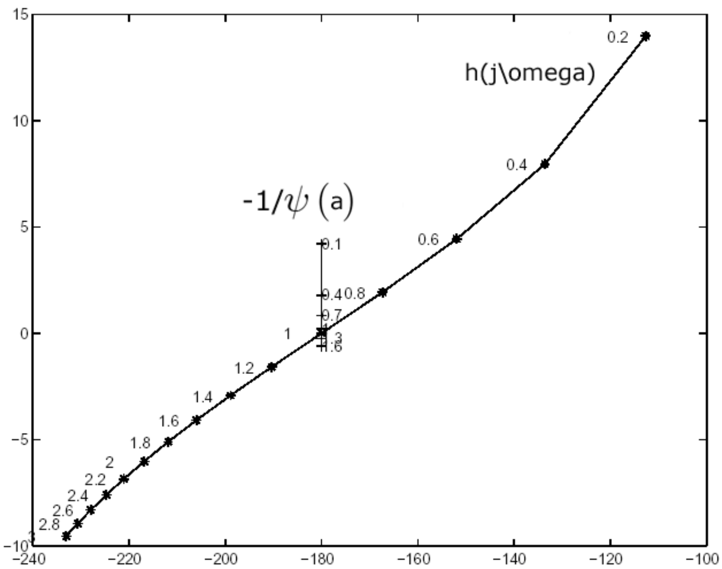

Figure 4: Nicols diagram of $h\left(j\omega\right)$ and $-\frac{1}{\psi(a)}$

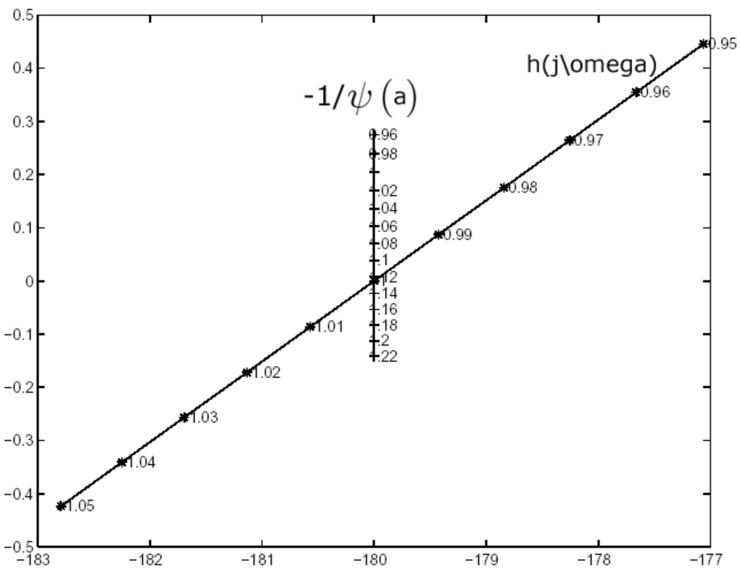

Figure 5: Nicols diagram of $h\left(j\omega\right)$ and $-\frac{1}{\psi(a)}$

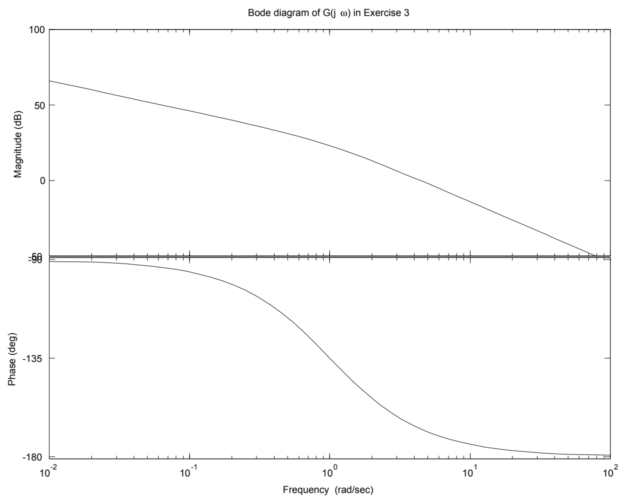

Figure 6: Bode plot of $G(j\omega)$

2. *Using $a\sin(\theta)$ as an argument in the nonlinearity it can be recognized that*

$$\psi(a\sin(\theta)) = \begin{cases} -L & \text{when } 0 \le \theta \le \alpha \text{ and } \pi+\alpha \le \theta \le 2\pi \\ L & \text{when } \alpha < \theta < \pi+\alpha \end{cases}$$

where

$$a\sin(\alpha) = S$$

Since $\psi(y)$ is not memoryless, the theory from Appendix A is applied. The describing function is derived according to

$$\begin{aligned} z_{1s} &= \int_0^{2\pi} \psi(a\sin(\theta))\sin(\theta)\,d\theta \\ &= \frac{1}{\pi}\int_0^{\alpha} -L\sin(\theta)\,d\theta + \frac{1}{\pi}\int_{\alpha}^{\pi+\alpha} L\sin(\theta)\,d\theta + \frac{1}{\pi}\int_{\pi+\alpha}^{2\pi} -L\sin(\theta)\,d\theta \\ &= \frac{1}{\pi}(L\cos\alpha - L) + \frac{1}{\pi}(2L\cos\alpha) + \frac{1}{\pi}(L + L\cos\alpha) \\ &= \frac{4L}{\pi}\cos\alpha \end{aligned}$$

and

$$\begin{aligned} z_{1c} &= \int_0^{2\pi} \psi\left(a\sin\left(\theta\right)\right)\cos\left(\theta\right) d\theta \\ &= \frac{1}{\pi}\int_0^{\alpha} -L\cos\left(\theta\right) d\theta + \frac{1}{\pi}\int_{\alpha}^{\pi+\alpha} L\cos\left(\theta\right) d\theta + \frac{1}{\pi}\int_{\pi+\alpha}^{2\pi} -L\cos\left(\theta\right) d\theta \\ &= \frac{1}{\pi}\left(-L\sin\alpha\right) + \frac{1}{\pi}\left(-2L\sin\alpha\right) + \frac{1}{\pi}\left(-L\sin\alpha\right) \\ &= -\frac{4L}{\pi}\sin\alpha \end{aligned}$$

and

$$\begin{aligned} z_1 &= \sqrt{z_{1s}^2 + z_{1c}^2} \\ &= \sqrt{\left(\frac{4L}{\pi}\cos\alpha\right)^2 + \left(-\frac{4L}{\pi}\sin\alpha\right)^2} \\ &= \sqrt{\frac{16L^2}{\pi^2}\cos^2\alpha + \frac{16L^2}{\pi^2}\sin^2\alpha} \\ &= \frac{4L}{\pi}\sqrt{\cos^2\alpha + \sin^2\alpha} \\ &= \frac{4L}{\pi} \end{aligned}$$

and

$$\begin{aligned} \varphi &= \arctan\left(\frac{z_{1c}}{z_{1s}}\right) \\ &= \arctan\left(\frac{-\frac{4L}{\pi}\sin\alpha}{\frac{4L}{\pi}\cos\alpha}\right) \\ &= \arctan\left(-\frac{\sin\alpha}{\cos\alpha}\right) \\ &= \arctan\left(-\tan\left(\alpha\right)\right) \\ &= \arctan\left(\tan\left(-\alpha\right)\right) \\ &= -\alpha \\ &= -\arcsin\left(\frac{S}{a}\right) \end{aligned}$$

Using the preceding calculations the describing function is given by

$$\begin{aligned}
\left|\Psi(a,\omega)\right| &= \frac{z_1}{a} \\
&= \frac{\frac{4L}{\pi}}{a} \\
&= \frac{4L}{\pi a}
\end{aligned}$$

and

$$\begin{aligned}
\angle\Psi(a,\omega) &= \varphi \\
&= -\arcsin\left(\frac{S}{a}\right)
\end{aligned}$$

3. *In order to draw* $-\frac{1}{\Psi(a,\omega)}$ *in a Nichols diagram as a function of* $\frac{a}{S}$*, we calculate* $\left|-\frac{1}{\Psi(a,\omega)}\right|$ *and* $\angle-\frac{1}{\Psi(a,\omega)}$ *as functions of* $\frac{a}{S}$

$$\begin{aligned}
\left|-\frac{1}{\Psi(a,\omega)}\right| &= \left|\frac{1}{\Psi(a,\omega)}\right| \\
&= \frac{1}{\left|\Psi(a,\omega)\right|} \\
&= \frac{1}{\frac{4L}{\pi a}} \\
&= \frac{\pi a}{4L} \\
&= \frac{\pi S}{4L}\left(\frac{a}{S}\right) \\
&\Rightarrow \frac{L}{S}\left|-\frac{1}{\Psi(a,\omega)}\right| = \frac{\pi}{4}\left(\frac{a}{S}\right)
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
\angle-\frac{1}{\Psi(a,\omega)} &= -180^\circ + \angle\frac{1}{\Psi(a,\omega)} \\
&= -180^\circ - \angle\Psi(a,\omega) \\
&= -180^\circ + \arcsin\left(\left(\frac{a}{S}\right)^{-1}\right)
\end{aligned} \tag{2}$$

Figure 7 shows a Nichols diagram of $-\frac{1}{\Psi(a,\omega)}$ *where the magnitude and phase are normalized with respect to* $\frac{a}{S}$*, resulting in scaled magnitudes.*

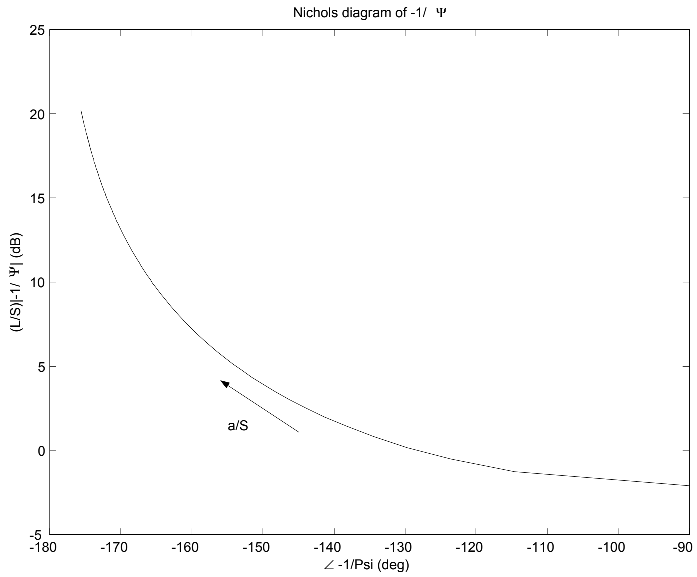

Figure 7: Nichols diagram of $-\frac{1}{\Psi}$ as a function of $\frac{a}{S}$

4. *Using the given constants the describing function is given by*

$$\begin{aligned} \left|-\frac{1}{\Psi(a,\omega)}\right| &= \frac{\pi}{4}a \\ \angle-\frac{1}{\Psi(a,\omega)} &= -180^\circ + \arcsin\left(a^{-1}\right) \end{aligned}$$

To investigate and estimate periodic solutions in the system, we first of all need to solve the harmonic balance equation

$$h(j\omega)\Psi(a,\omega) + 1 = 0$$

to establish existence of periodic solution. The harmonic balance equation can be reformulated as

$$h(j\omega) = -\frac{1}{\Psi(a,\omega)}$$

which is used to investigate periodic solutions in a Nichols diagram. From Figure 8 it can be seen that a periodic solution exists (the harmonic balance equation has a solution). By further investigation, estimates of frequency and amplitude are found as

$$\begin{aligned} \omega &\approx 3 \\ a &\approx 3 \end{aligned}$$

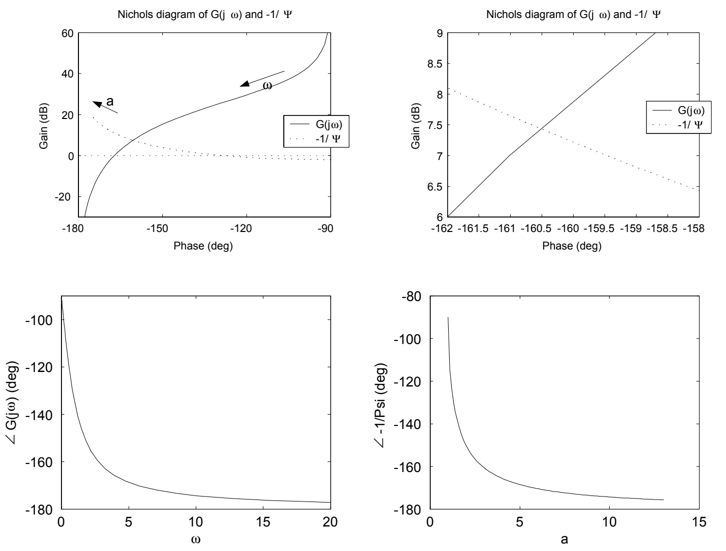

Figure 8: Graphical solution of the harmonic balance equation

5. *The periodic solution is stable. This can be established by investigating Figure 8. The periodic solutions imply that the temperature in the room will vary around a desired equilibrium with the given amplitude and frequency according to*

$$y(t) = y_0 + 3\sin(0.05t) \quad {}^\circ\text{C}$$

where t is in seconds (not in hours as the model).

6. *A simulation of the system is shown in Figure 9 where it can be seen that $a \approx 3$ and*

$$\begin{aligned} \Delta T &\approx 8 - 5.75 = 2.25 \\ &\Rightarrow f = \frac{1}{\Delta T} \approx 0.4 \\ &\Rightarrow \omega = 2\pi f \approx 2.79 \end{aligned}$$

which agrees with the results from the describing function method.

7. *Figure 10 shows a Nichols diagram of $G(j\omega)$ and $-\frac{1}{\Psi}$ where Ψ is expressed as a function of $\frac{a}{S}$. It can be recognized that there are several possibilities of reducing the amplitude a*

   - *moving $-\frac{1}{\Psi}$ to the left by reducing S (reducing S will only influence the phase, see (1) and (2))*

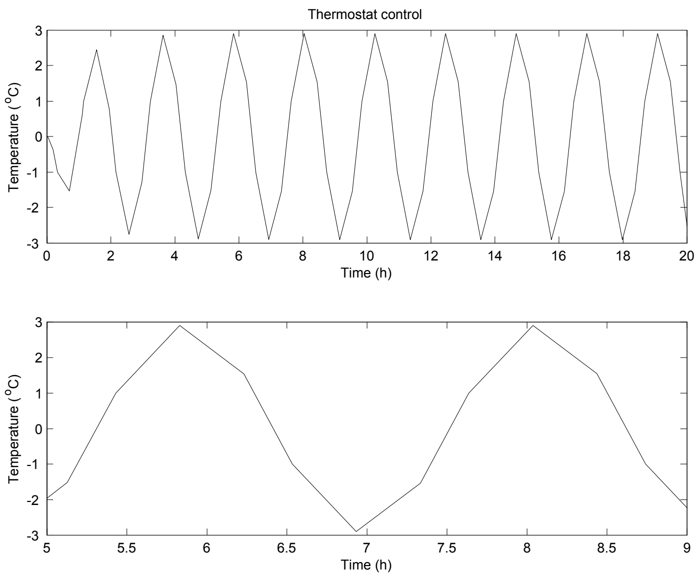

Figure 9: Simulation of the thermostat control system

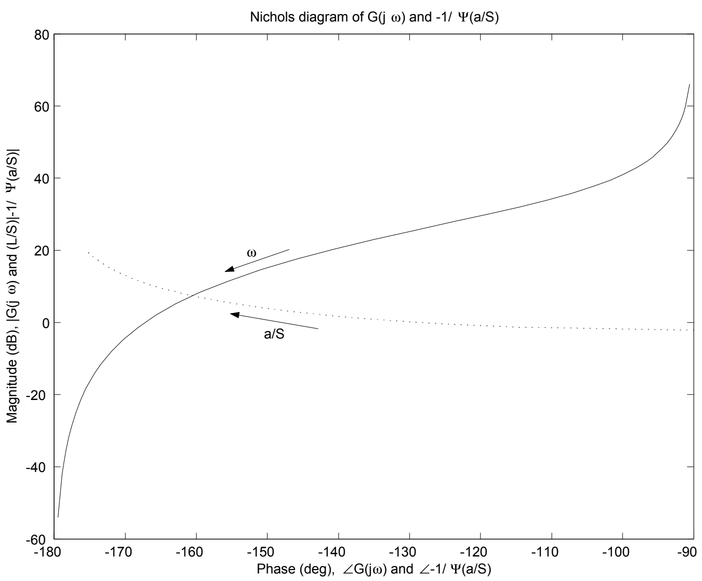

Figure 10: Nichols diagram of $G(j\omega)$ and $-\frac{1}{\Psi}$ when Ψ is expressed as a function of $\frac{a}{S}$

- *moving* $-\frac{1}{\Psi}$ *higher by reducing* L *(reducing* L *will only influence the magnitude, see (1) and (2))*
- *moving* $G(j\omega)$ *lower by reducing* K