TTK4150 Nonlinear Control Systems
Solution 5
Part 1

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Solution 1
The system is given by

$$h_{r1}(s) = K_p \frac{(1+T_i s)(1+T_d s)}{T_i s(1+\alpha T_d s)}$$

where $K_p > 0$, $0 \leq T_d < T_i$ *and* $0 \leq \alpha \leq 1$. *The system is analyzed in three steps due to* $0 \leq T_d$ *and* $0 \leq \alpha$ *(these constants may be zero). These steps consists of analyzing three systems*

$$h_{r11}(s) = K_p \frac{(1+T_i s)}{T_i s} \text{ where } T_d = 0 \quad (1)$$

$$h_{r12}(s) = K_p \frac{(1+T_i s)(1+T_d s)}{T_i s} \text{ where } T_d \neq 0 \text{ and } \alpha = 0 \quad (2)$$

$$h_{r13}(s) = K_p \frac{(1+T_i s)(1+T_d s)}{T_i s(1+\alpha T_d s)} \text{ where } T_d \neq 0 \text{ and } \alpha \neq 0 \quad (3)$$

The case in (1) is reformulated as

$$\begin{aligned} h_{r11}(s) &= K_p \frac{T_i\left(\frac{1}{T_i} + s\right)}{T_i s} \\ &= K_p \frac{(z_1 + s)}{s} \end{aligned}$$

where K_p *and* z_1 *are positive real constants. Further, since* K_p *is positive the passivity properties of* $h_{r11}(s) = K_p \frac{(z_1+s)}{s}$ *is the same as* $h_{r11'}(s) =$

$\frac{(z_1+s)}{s}$. *Passivity of* $h_{r11'}$ *is proved by using Proposition 1. The complex value* $h_{r11'}(j\omega)$ *is found as*

$$\begin{aligned} h_{r11'}(j\omega) &= \frac{(z_1 + j\omega)}{j\omega} \\ &= \frac{jz_1 - \omega}{-\omega} \\ &= 1 - j\frac{z_1}{\omega} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re}\left[h_{r11'}(j\omega)\right] &= 1 \\ &\geq 0 \ \forall\omega \end{aligned} \tag{4}$$

The second case, (2), is analyzed in the same manner. That is

$$\begin{aligned} h_{r12}(s) &= K_p\frac{(1+T_i s)(1+T_d s)}{T_i s} \\ &= K_p\frac{T_i\left(\frac{1}{T_i}+s\right)T_d\left(\frac{1}{T_d}+s\right)}{T_i s} \\ &= K_p T_d\frac{\left(\frac{1}{T_i}+s\right)\left(\frac{1}{T_d}+s\right)}{s} \\ &= K\frac{(z_1+s)(z_2+s)}{s} \end{aligned}$$

where K, z_1, *and* z_2 *are positive real constants. As in the previous case we continue our analyses on the system* $h_{r12'}(s) = \frac{(z_1+s)(z_2+s)}{s}$. *Passivity of* $h_{r12'}$ *is proved by using Proposition 1. The complex value* $h_{r12'}(j\omega)$ *is found as*

$$\begin{aligned} h_{r12'}(j\omega) &= \frac{(z_1+j\omega)(z_2+j\omega)}{j\omega} \\ &= \frac{j(z_1+j\omega)(z_2+j\omega)j}{-\omega} \\ &= \frac{j\left(j\omega z_1 + j\omega z_2 + z_1 z_2 - \omega^2\right)}{-\omega} \\ &= \frac{-\omega z_1 - \omega z_2 + jz_1 z_2 - j\omega^2}{-\omega} \\ &= (z_1+z_2) - j\frac{(z_1 z_2 - \omega^2)}{\omega} \end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}\left[h_{r12'}(j\omega)\right] &= z_1 + z_2 \\ &\geq 0 \ \forall\omega \end{aligned} \tag{5}$$

The third case, (3), is analyzed in the same manner. That is

$$\begin{aligned}h_{r13}(s) &= K_p \frac{(1+T_i s)(1+T_d s)}{T_i s(1+\alpha T_d s)} \\ &= K_p \frac{T_i\left(\frac{1}{T_i}+s\right) T_d\left(\frac{1}{T_d}+s\right)}{T_i s \alpha T_d\left(\frac{1}{\alpha T_d}+s\right)} \\ &= \frac{K_p}{\alpha} \frac{\left(\frac{1}{T_i}+s\right)\left(\frac{1}{T_d}+s\right)}{s\left(\frac{1}{\alpha T_d}+s\right)} \\ &= K \frac{(z_1+s)(z_2+s)}{s(p_1+s)}\end{aligned}$$

where K, z_1, z_2 and p_1 are positive real constants. As in the previous case we continue our analyses on the system $h_{r13'}(s) = \frac{(z_1+s)(z_2+s)}{s(p_1+s)}$. Passivity of $h_{r13'}$ is proved by using Proposition 1. The complex value $h_{r13'}(j\omega)$ is found as

$$\begin{aligned}h_{r13'}(j\omega) &= \frac{(z_1+j\omega)(z_2+j\omega)}{j\omega(p_1+j\omega)} \\ &= \frac{j(z_1+j\omega)(z_2+j\omega)(p_1-j\omega)}{-\omega(p_1^2+\omega^2)} \\ &= \frac{j\left(j\omega p_1 z_1 + j\omega p_1 z_2 - j\omega z_1 z_2 + p_1 z_1 z_2 + j\omega^3 - \omega^2 p_1 + \omega^2 z_1 + \omega^2 z_2\right)}{-\omega(p_1^2+\omega^2)} \\ &= \frac{-\omega p_1 z_1 - \omega p_1 z_2 + \omega z_1 z_2 + j p_1 z_1 z_2 - \omega^3 - j\omega^2 p_1 + j\omega^2 z_1 + j\omega^2 z_2}{-\omega(p_1^2+\omega^2)} \\ &= \frac{p_1 z_1 + p_1 z_2 - z_1 z_2 + \omega^2}{(p_1^2+\omega^2)} - j\frac{p_1 z_1 z_2 - \omega^2 p_1 + \omega^2 z_1 + \omega^2 z}{\omega(p_1^2+\omega^2)}\end{aligned}$$

and

$$
\begin{aligned}
\operatorname{Re}\left[h_{r13'}(j\omega)\right] &= \frac{p_1 z_1 + p_1 z_2 - z_1 z_2 + \omega^2}{(p_1^2 + \omega^2)} \\
&\geq \frac{p_1 z_1 + p_1 z_2 - z_1 z_2}{(p_1^2 + \omega^2)} \\
&= \frac{1}{(p_1^2 + \omega^2)} (p_1 z_1 + p_1 z_2 - z_1 z_2) \\
&= \frac{1}{(p_1^2 + \omega^2)} \left(\frac{1}{\alpha T_d} \frac{1}{T_i} + \frac{1}{\alpha T_d} \frac{1}{T_d} - \frac{1}{T_i} \frac{1}{T_d} \right) \\
&= \frac{1}{(p_1^2 + \omega^2)} \left(\frac{1}{\alpha T_d T_i} + \frac{1}{\alpha T_d^2} - \frac{\alpha}{\alpha T_i T_d} \right) \\
&= \frac{1}{(p_1^2 + \omega^2)\,\alpha T_d^2} + \frac{1}{(p_1^2 + \omega^2)} \left(\frac{1 - \alpha}{\alpha T_d T_i} \right) \\
&\geq \frac{1}{(p_1^2 + \omega^2)\,\alpha T_d^2} \\
&\geq 0 \ \forall \omega
\end{aligned} \tag{6}
$$

Using (4), (5) and (6) we conclude by Proposition 1 that the PID control law h_{r1} is passive.

Solution 2
The system is given by

$$
h_{r2}(s) = K_p \beta \frac{(1 + T_i s)(1 + T_d s)}{(1 + \beta T_i s)(1 + \alpha T_d s)}
$$

where $K_p > 0$, $0 \leq T_d < T_i$, $0 < \alpha \leq 1$ and $1 \leq \beta \leq \infty$. Analysis is done in two steps due to $0 \leq T_d$ (the constant may be zero). These steps consists of evaluating two systems

$$
\begin{aligned}
h_{r21}(s) &= K_p \beta \frac{(1 + T_i s)}{(1 + \beta T_i s)} \text{ where } T_d = 0 \\
h_{r22}(s) &= K_p \beta \frac{(1 + T_i s)(1 + T_d s)}{(1 + \beta T_i s)(1 + \alpha T_d s)} \text{ where } T_d \neq 0
\end{aligned}
$$

which both have poles with real parts less than zero. Hence, we will apply Theorem 1 to study the passivity properties of the control law.

1. *For the first system we have that*

$$\begin{aligned}
|h_{r21}(j\omega)| &= \left|K_p\beta\frac{1+jT_i\omega}{1+j\beta T_i\omega}\right| \\
&= |K_p\beta|\left|\frac{1+jT_i\omega}{1+j\beta T_i\omega}\right| \\
&= K_p\beta\frac{|1+jT_i\omega|}{|1+j\beta T_i\omega|} \\
&\leq K_p\beta
\end{aligned}$$

where we have used $\beta \geq 1$. For $h_{r21}(s)$ it can be seen that choosing $T_d = 0$ is the same as choosing $\alpha = 1$. Using this we can upper bound $|h_{r21}(j\omega)|$ as

$$|h_{r21}(j\omega)| \leq \frac{K_p\beta}{\alpha} \tag{7}$$

An upper bound on the magnitude of $h_{r22}(j\omega)$ is found as

$$\begin{aligned}
|h_{r22}(j\omega)| &= \left|K_p\beta\frac{(1+T_ij\omega)(1+jT_d\omega)}{(1+j\beta T_i\omega)(1+j\alpha T_d\omega)}\right| \\
&= |K_p\beta|\left|\frac{1+T_ij\omega}{1+j\beta T_i\omega}\right|\left|\frac{1+jT_d\omega}{1+j\alpha T_d\omega}\right| \\
&= K_p\beta\left|\frac{1+T_ij\omega}{1+j\beta T_i\omega}\right|\left|\frac{1+jT_d\omega}{\alpha\left(\frac{1}{\alpha}+jT_d\omega\right)}\right| \\
&= K_p\beta\left|\frac{1}{\alpha}\right|\left|\frac{1+T_ij\omega}{1+j\beta T_i\omega}\right|\left|\frac{1+jT_d\omega}{\frac{1}{\alpha}+jT_d\omega}\right| \\
&= \frac{K_p\beta}{\alpha}\left|\frac{1+T_ij\omega}{1+j\beta T_i\omega}\right|\left|\frac{1+jT_d\omega}{\frac{1}{\alpha}+jT_d\omega}\right|
\end{aligned}$$

where

$$\begin{aligned}
\left|\frac{1+T_ij\omega}{1+j\beta T_i\omega}\right| &= \frac{1+T_i^2\omega^2}{1+\beta^2T_i^2\omega^2} \\
&\leq 1
\end{aligned}$$

since $\beta \geq 1$, and

$$\begin{aligned}
\left|\frac{1+jT_d\omega}{\frac{1}{\alpha}+jT_d\omega}\right| &= \frac{1+T_d^2\omega^2}{\left(\frac{1}{\alpha}\right)^2+T_d^2\omega^2} \\
&\leq 1
\end{aligned}$$

since $\left(\frac{1}{\alpha}\right) \geq 1$. *An upper bound on* $|h_{r22}(j\omega)|$ *is therefore given by*

$$|h_{r22}(j\omega)| = \frac{K_p\beta}{\alpha} \tag{8}$$

Using (7)-(8) we conclude that

$$h_{r2}(s) = K_p\beta\frac{(1+T_is)(1+T_ds)}{(1+\beta T_is)(1+\alpha T_ds)}$$

satisfies

$$|h_{r2}(j\omega)| = \frac{K_p\beta}{\alpha} \tag{9}$$

given the conditions stated in Proposition 3.

2. *For the first system we have that*

$$\begin{aligned} h_{r21}(j\omega) &= K_p\beta\frac{(1+jT_i\omega)}{(1+j\beta T_i\omega)} \\ &= K_p\beta\frac{(1+jT_i\omega)(1-j\beta T_i\omega)}{1+\beta^2T_i^2\omega^2} \\ &= K_p\beta\frac{j\omega T_i - j\beta\omega T_i + \beta\omega^2T_i^2+1}{1+\beta^2T_i^2\omega^2} \\ &= K_p\beta\frac{(\beta\omega^2T_i^2+1)+j(\omega T_i-\beta\omega T_i)}{1+\beta^2T_i^2\omega^2} \end{aligned}$$

where it can be recognized that

$$\begin{aligned} \mathrm{Re}\left[h_{r21}(j\omega)\right] &= K_p\beta\frac{\beta\omega^2T_i^2+1}{1+\beta^2T_i^2\omega^2} \\ &= K_p\frac{\beta T_i^2\omega^2+1}{\beta T_i^2\omega^2+\frac{1}{\beta}} \\ &\geq K_p \end{aligned} \tag{10}$$

In the second case we have that

$$\begin{aligned} h_{r22}(j\omega) &= K_p\beta\frac{(1+T_ij\omega)(1+T_dj\omega)}{(1+\beta T_ij\omega)(1+\alpha T_dj\omega)} \\ &= K_p\beta\frac{(1+T_ij\omega)(1+T_dj\omega)(1-\beta T_ij\omega)(1-\alpha T_dj\omega)}{\left(1+\beta^2T_i^2\omega^2\right)(1+\alpha^2T_d^2\omega^2)} \end{aligned}$$

where the numerator is calculated as

$$
\begin{aligned}
(1+T_i j\omega)(1+T_d j\omega)(1-\beta T_i j\omega)(1-\alpha T_d j\omega) &= j\omega T_d + j\omega T_i - j\alpha\omega T_d - j\beta\omega T_i + j^2\omega^2 T_d T_i \\
&\quad -j^2\alpha\omega^2 T_d T_i - j^2\beta\omega^2 T_d T_i + j^2\alpha\beta\omega^2 T_d T_i \\
&\quad -j^2\alpha\omega^2 T_d^2 - j^2\beta\omega^2 T_i^2 - j^3\alpha\omega^3 T_d^2 T_i \\
&\quad -j^3\beta\omega^3 T_d T_i^2 + j^3\alpha\beta\omega^3 T_d T_i^2 + j^3\alpha\beta\omega^3 T_d^2 T_i \\
&\quad +j^4\alpha\beta\omega^4 T_d^2 T_i^2 + 1 \\
&= \left(\begin{array}{c} -\omega^2 T_d T_i + \alpha\omega^2 T_d T_i + \beta\omega^2 T_d T_i \\ -\alpha\beta\omega^2 T_d T_i + \alpha\omega^2 T_d^2 + \beta\omega^2 T_i^2 \\ +\alpha\beta\omega^4 T_d^2 T_i^2 + 1 \end{array}\right) \\
&\quad +j\left(\begin{array}{c} \omega T_d + \omega T_i - \alpha\omega T_d - \beta\omega T_i \\ +\alpha\omega^3 T_d^2 T_i + \beta\omega^3 T_d T_i^2 \\ -\alpha\beta\omega^3 T_d T_i^2 - \alpha\beta\omega^3 T_d^2 T_i \end{array}\right)
\end{aligned}
$$

The real value of $h_{r22}(j\omega)$ is now found as

$$
\begin{aligned}
\operatorname{Re}[h_{r22}(j\omega)] &= K_p\beta\frac{\left(\begin{array}{c} -\omega^2 T_d T_i + \alpha\omega^2 T_d T_i + \beta\omega^2 T_d T_i - \alpha\beta\omega^2 T_d T_i \\ +\alpha\omega^2 T_d^2 + \beta\omega^2 T_i^2 + \alpha\beta\omega^4 T_d^2 T_i^2 + 1 \end{array}\right)}{\left(1+\beta^2 T_i^2\omega^2\right)\left(1+\alpha^2 T_d^2\omega^2\right)} \\
&= K_p\frac{\beta\left(\begin{array}{c} -\omega^2 T_d T_i + \alpha\omega^2 T_d T_i + \beta\omega^2 T_d T_i - \alpha\beta\omega^2 T_d T_i \\ +\alpha\omega^2 T_d^2 + \beta\omega^2 T_i^2 + \alpha\beta\omega^4 T_d^2 T_i^2 + 1 \end{array}\right)}{\left(1+\beta^2 T_i^2\omega^2\right)\left(1+\alpha^2 T_d^2\omega^2\right)} \\
&= K_p\frac{n(\omega)}{d(\omega)} \\
&= K_p\frac{d(\omega)+\gamma(\omega)}{d(\omega)}
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma(\omega) &= n(\omega) - d(\omega) \\
&= \beta\left(-\omega^2 T_d T_i + \alpha\omega^2 T_d T_i + \beta\omega^2 T_d T_i - \alpha\beta\omega^2 T_d T_i + \alpha\omega^2 T_d^2 + \beta\omega^2 T_i^2 + \alpha\beta\omega^4 T_d^2 T_i^2 + 1\right) \\
&\quad -\left(1+\beta^2 T_i^2\omega^2\right)\left(1+\alpha^2 T_d^2\omega^2\right) \\
&= \beta - \beta\omega^2 T_d T_i + \alpha\beta\omega^2 T_d T_i + \alpha\beta\omega^2 T_d^2 + \beta^2\omega^2 T_d T_i - \alpha\beta^2\omega^2 T_d T_i - \alpha^2\omega^2 T_d^2 \\
&\quad +\alpha\beta^2\omega^4 T_d^2 T_i^2 - \alpha^2\beta^2\omega^4 T_d^2 T_i^2 - 1
\end{aligned}
$$

The real value of $h(j\omega)$ may now be rewritten as

$$
\operatorname{Re}[h_{r22}(j\omega)] = K_p\frac{\left(1+\beta^2 T_i^2\omega^2\right)\left(1+\alpha^2 T_d^2\omega^2\right)+\gamma(\omega)}{\left(1+\beta^2 T_i^2\omega^2\right)\left(1+\alpha^2 T_d^2\omega^2\right)}
$$

and it can be recognized that to prove

$$\mathrm{Re}\left[h_{r22}\left(j\omega\right)\right] \geq K_p$$

is the same as proving

$$\gamma\left(\omega\right) \geq 0$$

This is done as

$$\begin{aligned}
\gamma\left(\omega\right) &= \beta - \beta\omega^2 T_d T_i + \alpha\beta\omega^2 T_d T_i + \alpha\beta\omega^2 T_d^2 + \beta^2\omega^2 T_d T_i - \alpha\beta^2\omega^2 T_d T_i - \alpha^2\omega^2 T_d^2 \\
&\quad + \alpha\beta^2\omega^4 T_d^2 T_i^2 - \alpha^2\beta^2\omega^4 T_d^2 T_i^2 - 1 \\
&= \left(\beta - 1\right) + \left(-\beta\omega^2 + \alpha\beta\omega^2 + \beta^2\omega^2 - \alpha\beta^2\omega^2\right) T_d T_i + \left(\alpha\beta\omega^2 - \alpha^2\omega^2\right) T_d^2 \\
&\quad + \left(\alpha\beta^2\omega^4 - \alpha^2\beta^2\omega^4\right) T_d^2 T_i^2 \\
&= \left(\beta - 1\right) + \left(\beta - 1 + \alpha - \alpha\beta\right)\beta\omega^2 T_d T_i + \left(\beta - \alpha\right)\alpha\omega^2 T_d^2 + \left(1 - \alpha\right)\alpha\beta^2\omega^4 T_d^2 T_i^2 \\
&= \left(\beta - 1\right) + \left(\beta - 1\right)\left(1 - \alpha\right)\beta\omega^2 T_d T_i + \left(\beta - \alpha\right)\alpha\omega^2 T_d^2 + \left(1 - \alpha\right)\alpha\beta^2\omega^4 T_d^2 T_i^2 \\
&= \underbrace{\left(\beta - 1\right)}_{\geq 0} + \underbrace{\left(\beta - 1\right)}_{\geq 0}\underbrace{\left(1 - \alpha\right)}_{\geq 0}\beta\omega^2 T_d T_i + \underbrace{\left(\beta - \alpha\right)}_{\geq 0}\alpha\omega^2 T_d^2 + \underbrace{\left(1 - \alpha\right)}_{\geq 0}\alpha\beta^2\omega^4 T_d^2 T_i^2 \\
&\geq 0
\end{aligned}$$

by which we conclude that

$$\mathrm{Re}\left[h_{r22}\left(j\omega\right)\right] \geq K_p \tag{11}$$

Using (10)-(11) we conclude that

$$h_{r2}\left(s\right) = K_p\beta\frac{\left(1 + T_i s\right)\left(1 + T_d s\right)}{\left(1 + \beta T_i s\right)\left(1 + \alpha T_d s\right)}$$

satisfies

$$\mathrm{Re}\left[h_{r2}\left(j\omega\right)\right] \geq K_p \tag{12}$$

given the conditions stated in Proposition 3.

3. *To prove that h_{r2} is passive is the same as proving* $\mathrm{Re}\left[h_{r2}\left(j\omega\right)\right] \geq 0\ \forall\omega$ *(Theorem 1). Since* $\mathrm{Re}\left[h_{r2}\left(j\omega\right)\right] \geq K_p > 0\ \forall\omega$ *we conclude that the control law is passive.*

4. *To prove that h_{r2} is input strictly passive is the same as proving that* $\mathrm{Re}\left[h_{r2}\left(j\omega\right)\right] \geq \delta \geq 0\ \forall\omega$ *(Theorem 1) for some positive δ. Since* $\mathrm{Re}\left[h_{r2}\left(j\omega\right)\right] \geq K_p > 0\ \forall\omega$ *we conclude that the control law is input strictly passive.*

5. *To prove that the system is output strictly passive is the same as proving that* $\text{Re}\left[h_{r2}\left(j\omega\right)\right] \geq \varepsilon \left|h_{r2}\left(j\omega\right)\right|^2 \ \forall\omega$ *for some positive* ε*. From (9) and (12) we know that*

$$\begin{aligned} \left|h_{r2}\left(j\omega\right)\right| &\leq \frac{K_p\beta}{\alpha} \\ \text{Re}\left[h_{r2}\left(j\omega\right)\right] &\geq K_p \end{aligned}$$

Using these inequalities, an upper bound on $\left|h_{r2}\left(j\omega\right)\right|^2$ *is found as*

$$\begin{aligned} \left|h_{r2}\left(j\omega\right)\right|^2 &\leq \left(\frac{K_p\beta}{\alpha}\right)^2 \\ &= \frac{K_p\beta^2}{\alpha^2}K_p \\ &\leq \frac{K_p\beta^2}{\alpha^2}\text{Re}\left[h_{r2}\left(j\omega\right)\right] \end{aligned}$$

which is rewritten as

$$\begin{aligned} \text{Re}\left[h_{r2}\left(j\omega\right)\right] &\geq \frac{\alpha^2}{K_p\beta^2}\left|h_{r2}\left(j\omega\right)\right|^2 \\ &= \varepsilon\left|h_{r2}\left(j\omega\right)\right|^2 \end{aligned}$$

and output strictly passivity of the control law is concluded.

6. *The system is given by*

$$h_{r2}\left(s\right) = \frac{u\left(s\right)}{e\left(s\right)} = K_p\beta\frac{\left(1+T_is\right)\left(1+T_ds\right)}{\left(1+\beta T_is\right)\left(1+\alpha T_ds\right)}$$

where e *is the input and* u *is the output. When investigating if a system is zero-state observable, the system is analyzed with inputs set to zero,* $e = 0$*. This leads to the equation*

$$\begin{aligned} \frac{u\left(s\right)}{e\left(s\right)} &= K_p\beta\frac{\left(1+T_is\right)\left(1+T_ds\right)}{\left(1+\beta T_is\right)\left(1+\alpha T_ds\right)} \\ &\Leftrightarrow\ u\left(s\right)\left(1+\beta T_is\right)\left(1+\alpha T_ds\right) = K_p\beta\left(1+T_is\right)\left(1+T_ds\right)e\left(s\right) \\ &\Rightarrow\ u\left(s\right)\left(1+\beta T_is\right)\left(1+\alpha T_ds\right) = 0 \text{ when } e\left(s\right) = 0 \\ &\Leftrightarrow\ u\left(s\right)\left(1+\beta T_is+\alpha T_ds+\beta T_i\alpha T_ds^2\right) = 0 \\ &\Leftrightarrow\ u+\beta T_i\dot{u}+\alpha T_d\dot{u}+\beta T_i\alpha T_d\ddot{u} = 0 \end{aligned}$$

Let $z_1 = u$, $z_2 = \dot{u}$ and $y = z_1$, then the control law with zero input can be expressed as

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{1}{\beta T_i \alpha T_d}\left(-z_1 - \left(\beta T_i + \alpha T_d\right) z_2\right) \\ y &= z_1\end{aligned}$$

To show that the system is zero-state observable we require that no solution can stay identical in $y = 0$ other than the trivial solution $z \equiv 0$. This is done as

$$\begin{aligned}y\left(t\right) &\equiv 0 \Leftrightarrow z_1\left(t\right) \equiv 0 \\ \dot{z}_1\left(t\right) &= 0 \Rightarrow z_2\left(t\right) \equiv 0 \\ \dot{z}_2 &= 0 \Rightarrow z_2 = \frac{1}{\left(\beta T_i + \alpha T_d\right)} z_1 = 0\end{aligned}$$

by which we conclude that the PID control law is zero-state observable.

Solution 3

See Theorem 6.3 in Khalil, and comment on necessary requirements.

Solution 4 (Exercise 6.11 in Khalil)

The system is given by

$$\begin{aligned}J_1\dot{\omega}_1 &= \left(J_2 - J_3\right)\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 &= \left(J_3 - J_1\right)\omega_3\omega_1 + u_2 \\ J_3\dot{\omega}_3 &= \left(J_1 - J_2\right)\omega_1\omega_2 + u_3\end{aligned}$$

where $u = [u_1\ u_2\ u_3]^T$ and $\omega = [\omega_1\ \omega_2\ \omega_3]$.

1. *Let $V\left(\omega\right) = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$ be a candidate for a storage function. The time derivative along the trajectories of the system is found as*

$$\begin{aligned}\dot{V}\left(\omega\right) &= J_1\dot{\omega}_1\omega_1 + J_2\dot{\omega}_2\omega_2 + J_3\dot{\omega}_3\omega_3 \\ &= \left(\left(J_2 - J_3\right)\omega_2\omega_3 + u_1\right)\omega_1 \\ &\quad + \left(\left(J_3 - J_1\right)\omega_3\omega_1 + u_2\right)\omega_2 \\ &\quad + \left(\left(J_1 - J_2\right)\omega_1\omega_2 + u_3\right)\omega_3 \\ &= \left(J_2 - J_3\right)\omega_1\omega_2\omega_3 + \left(J_3 - J_1\right)\omega_1\omega_2\omega_3 + \left(J_1 - J_2\right)\omega_1\omega_2\omega_3 \\ &\quad + u_1\omega_1 + u_2\omega_2 + u_3\omega_3 \\ &= \left(J_2 - J_3 + J_3 - J_1 + J_1 - J_2\right)\omega_1\omega_2\omega_3 + u_1\omega_1 + u_2\omega_2 + u_3\omega_3 \\ &= u^T\omega\end{aligned}$$

which shows that the map from u to ω is lossless with the storage function $V(\omega)$.

2. *With $u = -K\omega + v$ where $K = K^T$ where we have that*

$$\begin{aligned}
\dot{V}(\omega) &= u^T\omega \\
&= (-K\omega + v)^T \omega \\
&= -\omega^T K^T \omega + v^T\omega \\
&= v^T\omega - \omega^T K\omega \\
&\leq v^T\omega - \lambda_{\min}(K)\,\omega^T\omega \\
&\Rightarrow v^T\omega \geq \dot{V}(\omega) + \lambda_{\min}(K)\,\omega^T\omega
\end{aligned}$$

From the last equation it can be seen that the system is output strictly passive from v to ω with $v^T\omega \geq \dot{V}(\omega) + \lambda_{\min}(K)\,\omega^T\omega$. Hence, the map from v to ω is finite gain L_2 stable with L_2 gain less than or equal to $\frac{1}{\lambda_{\min}(K)}$ (Lemma 6.5).

3. *With $u = -K\omega$, we have that*

$$\dot{V}(\omega) \leq -\lambda_{\min}(K)\,\omega^T\omega$$

for the system $\dot{\omega} = f(\omega, -K\omega) = f'(\omega)$. Since $V(\omega)$ is positive definite and radially unbounded and $\dot{V}(\omega)$ is negative definite, we conclude that the system is globally asymptotically stable.

Solution 5 (Exercise 6.14 in Khalil)
Two systems

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - h_1(x_2) + e_1 \\ y_1 = x_2 \end{cases}$$

and

$$H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = h_2(x_3) \end{cases}$$

are connected as shown in Figure 6.11 in Khalil. The functions $h_i(\cdot)$ are locally Lipschitz and $h_i(\cdot) \in (0, \infty]$. Further, the function $h_2(z)$ satisfies $|h_2(z)| \geq \frac{|z|}{(1+z^2)}$.

1. *First the passivity properties of H_1 is investigated. Let $V_1(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ be a candidate for a storage function. The time derivative*

along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}_1(x_1,x_2) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1x_2 + x_2(-x_1 - h_1(x_2) + e_1) \\
&= x_1x_2 - x_1x_2 - h_1(x_2)x_2 + e_1x_2 \\
&= -h_1(x_2)x_2 + e_1x_2 \\
&= -h_1(y_1)y_1 + e_1y_1 \\
&\Rightarrow e_1y_1 = \dot{V}_1(x_1,x_2) + y_1h_1(y_1)
\end{aligned}$$

Since $h_1 \in (0,\infty]$, H_1 is output strictly passive. The passivity properties of H_2 is investigated by using $V_2(x_3) = \int_0^{x_3} h_2(z)\,dz$ as a candidate for a storage function. The time derivative along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}_2(x_3) &= \frac{\partial}{\partial x_3}\left(\int_0^{x_3} h_2(z)\,dz\right)\dot{x}_3 \\
&= h_2(x_3)(-x_3 + e_2) \\
&= -x_3h_2(x_3) + h_2(x_3)e_2 \\
&= -x_3h_2(x_3) + y_2e_2 \\
&\Rightarrow y_2e_2 = \dot{V}_2(x_3) + x_3h_2(x_3)
\end{aligned}$$

Since $h_2 \in (0,\infty]$, H_2 is strictly passive. By Theorem 6.1 we conclude that the feedback connection is passive.

2. *Asymptotic stability of the unforced system is shown by using Theorem 6.3. Since we have one strictly passive system and one output strictly passive system, we need to show that the system which is output strictly passive also is zero-state observable. It can be recognized that no solution can stay identical in $S = \{x_2 = 0\}$ other than the trivial solution $(x_1,x_2) = (0,0)$. That is*

$$\begin{aligned}
y_1 &\equiv 0 \Leftrightarrow x_2 \equiv 0 \\
\dot{x}_2 &= 0 \Rightarrow x_1 = -h_1(x_2) = 0
\end{aligned}$$

Hence, the unforced system is asymptotically stable. To prove global results, we need to show that the storage functions are radially unbounded. The first storage function is given by

$$\begin{aligned}
V_1(x_1,x_2) &= \frac{1}{2}\left(x_1^2 + x_2^2\right) \\
&= \frac{1}{2}\left\|(x_1,x_2)\right\|_2
\end{aligned}$$

which clearly is radially unbounded. The second storage function is given by

$$\begin{aligned} V_2(x_3) &= \int_0^{x_3} h_2(z)\,dz \\ &\geq \int_0^{x_3} \frac{|z|}{(1+z^2)} dz \\ &= \int_0^{x_3} \frac{z}{(1+z^2)} d \\ &= \frac{1}{2}\ln\left(1+x_3^2\right) \end{aligned}$$

where it can be recognized that $V_2(x_3) \to \infty$ as $|x_3| \to \infty$. Hence, the unforced system is globally asymptotically stable.

Solution 6 (Exercise 6.15 in Khalil)
Two systems

$$H_1 : \begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -x_1^3 - x_2 + e_1 \\ y_1 = x_2 \end{cases}$$

and

$$H_2 : \begin{cases} \dot{x}_3 = -x_3 + e_2 \\ y_2 = x_3^3 \end{cases}$$

are connected as shown in Figure 6.11 in Khalil.

1. *First the passivity properties of H_1 is investigated. Let $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ be a candidate for a storage function. The time derivative along the trajectories of the system is found as*

$$\begin{aligned} \dot{V}_1(x_1,x_2) &= x_1^3\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1^3(-x_1+x_2) + x_2\left(-x_1^3 - x_2 + e_1\right) \\ &= -x_1^4 + x_1^3x_2 - x_1^3x_2 - x_2^2 + x_2e_1 \\ &= -x_1^4 - x_2^2 + e_1y_1 \\ &\Rightarrow\ e_1y_1 = \dot{V}_1(x_1,x_2) + x_1^4 + x_2^2 \end{aligned}$$

Hence, H_1 is strictly passive with storage function $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. First the passivity properties of H_1 is investigated. Let $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ be a candidate for a storage function. The passivity properties of H_2 is investigated by using $V_2(x_3) = \frac{1}{4}x_3^4$ as a candidate for *a storage function. The time derivative along the trajectories of the system is found as*

$$\begin{aligned}\dot{V}_2(x_3) &= x_3^3\dot{x}_3 \\ &= x_3^3(-x_3+e_2) \\ &= -x_3^4+x_3^3e_2 \\ &= -x_3^4+e_2y_2\end{aligned}$$

Hence, H_2 is strictly passive with storage function $V_2(x_3)=\frac{1}{4}x_3^4$. It follows from Theorem 6.3 that the origin of the unforced system is asymptotically stable. Moreover, since V_1 and V_2 are radially unbounded we can conclude that the results holds globally.