TTK4150 Nonlinear Control Systems
Solution 4

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Solution 1

1. *The function* $V_1(x_1, x_2 t)$ *is given by*

$$V_1(x_1, x_2 t) = x_1^2 + \left(1 + e^t\right) x_2^2$$

Since $e^t \to \infty$ *when* $t \to \infty$, *the term* $(1 + e^t)$ *may not be upper bounded uniformly in* t. *Hence, the function is not decresent. The function may however be lower bounded by*

$$\begin{aligned} V_1(x_1, x_2 t) &= x_1^2 + \left(1 + e^t\right) x_2^2 \\ &\geq x_1^2 + x_2^2 \\ &= W_1(x) \end{aligned}$$

where it can be recognized that $W_1(x)$ *is positive definite. This implies that the function* $V_1(x_1, x_2 t)$ *is positive definite.*

2. *The function* $V_2(x_1, x_2 t)$ *is given by*

$$V_2(x_1, x_2 t) = \frac{x_1^2 + x_2^2}{1 + t}$$

Since $\frac{1}{1+t} \to 0$ *when* $t \to \infty$ *the function* $V_2(x_1, x_2 t)$ *may not be lower bounded uniformly in* t. *Hence, the function is not positive definite. The function may however be upper bounded by*

$$\begin{aligned} V_2(x_1, x_2 t) &= \frac{x_1^2 + x_2^2}{1 + t} \\ &\leq x_1^2 + x_2^2 \\ &= W_2(x) \end{aligned}$$

where it can be recognized that $W_2(x)$ *is positive definite. This implies that the function* $V_2(x_1, x_2 t)$ *is decresent.*

3. *The function* $V_3(x_1, x_2 t)$ *is given by*

$$V_3(x_1, x_2 t) = \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right)$$

The function may be lower and upper bounded according to

$$\begin{aligned} V_3(x_1, x_2 t) &= \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right) \\ &\geq x_1^2 + x_2^2 \\ &= W_1(x) \end{aligned}$$

and

$$\begin{aligned} V_3(x_1, x_2 t) &= \left(1 + \cos^4 t\right)\left(x_1^2 + x_2^2\right) \\ &\leq 2\left(x_1^2 + x_2^2\right) \\ &= W_2(x) \end{aligned}$$

Since $W_1(x)$ *and* $W_2(x)$ *are both positive definite, we conclude that the function* $V_3(x_1, x_2 t)$ *is positive definite and decresent.*

Solution 2

The system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 - g(t)\, x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

A Lyapunov function candidate is taken as

$$V(t, x) = x_1^2 + (1 + g(t))\, x_2^2$$

By using $0 \leq g(t) \leq k$ *we see that*

$$\begin{aligned} V(t, x) &= x_1^2 + (1 + g(t))\, x_2^2 \\ &\leq x_1^2 + (1 + k)\, x_2^2 \\ &\leq \|x\|_2^2 \\ &= W_2(x) \end{aligned} \tag{1}$$

and

$$\begin{aligned} V(t, x) &= x_1^2 + (1 + g(t))\, x_2^2 \\ &\geq x_1^2 + x_2^2 \\ &= \|x\|_2^2 \\ &= W_1(x) \end{aligned} \tag{2}$$

where it can be recognized that $W_1(x)$ *and* $W_2(x)$ *are positive definite, implying that* $V(t,x)$ *is positive definite and decresent. The time derivative of the Lyapunov function candidate along the trajectories of the system is found as*

$$\begin{aligned}
\dot{V}(t,x) &= 2x_1\dot{x}_1 + \dot{g}(t)x_2^2 + 2(1+g(t))x_2\dot{x}_2 \\
&= 2x_1(-x_1 - g(t)x_2) + \dot{g}(t)x_2^2 + 2(1+g(t))x_2(x_1 - x_2) \\
&= -2x_1^2 - 2g(t)x_1x_2 + \dot{g}(t)x_2^2 + 2x_1x_2 - 2x_2^2 + 2g(t)x_2x_1 - 2g(t)x_2^2 \\
&= -2x_1^2 + \dot{g}(t)x_2^2 + 2x_1x_2 - 2x_2^2 - 2g(t)x_2^2 \\
&= -2x_1^2 + 2x_1x_2 - 2x_2^2 - (2g(t) - \dot{g}(t))x_2^2
\end{aligned}$$

It can be recognized that $(2g(t) - \dot{g}(t)) \geq 0$ *since* $\dot{g}(t) \leq g(t)$. *Using this, the time derivative of* $V(t,x)$ *is upper bounded by*

$$\begin{aligned}
\dot{V}(t,x) &= -2x_1^2 + 2x_1x_2 - 2x_2^2 - (2g(t) - \dot{g}(t))x_2^2 \\
&\leq -2x_1^2 + 2x_1x_2 - 2x_2^2 \\
&= -x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x \\
&= -x^T Q x
\end{aligned}$$

where it can be recognized that Q *is positive definite,* $eig(Q) = \{1,3\}$. *This implies that*

$$\begin{aligned}
\dot{V}(t,x) &\leq -x^T Q x \\
&\leq \|x\|_2^2
\end{aligned} \tag{3}$$

Using (1), (2) and (3) we conclude by Theorem 4.10 that the origin is globally exponentially stable.

Solution 3

The system is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - c(t)x_2
\end{aligned}$$

A Lyapunov function candidate is taken as

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right) \tag{4}$$

The time derivative of $V(x)$ along the trajectories of the system is found as

$$\begin{aligned} \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2 + x_2(-x_1 - c(t)x_2) \\ &= x_1x_2 - x_1x_2 - c(t)x_2^2 \\ &= -c(t)x_2^2 \\ &\leq -k_1x_2^2 \end{aligned} \tag{5}$$

and it can be seen that $\dot{V}(x)$ is negative semidefinite. By Theorem 4.8 we conclude that the origin is uniformly stable ($V(x)$ is positive definite and decresent). In order to prove that $x_2 \to 0$ as $t \to \infty$ we apply Barbalat's lemma. Since $\dot{V}(x) = -c(t)x_2^2$ where $c(t)$ is some bounded value greater than zero, $\dot{V}(x) = 0 \Leftrightarrow x_2 = 0$. Following the notation of Lemma 8.2, let $\phi(t) = \dot{V}(t)$. $\dot{V}(t)$ is uniformly continuous in t if $\ddot{V}(t)$ is bounded

$$\begin{aligned} \ddot{V}(t) &= -\dot{c}(t)x_2^2 - 2c(t)x_2\dot{x}_2 \\ &= -\dot{c}(t)x_2^2 - 2c(t)x_2(-x_1 - c(t)x_2) \\ &= -\dot{c}(t)x_2^2 - 2c(t)x_1x_2 - 2c^2(t)x_2^2 \end{aligned}$$

Using (4) and (5) it can be recognized that $V(t) \leq V(t_0)$, which implies that x_1 and x_2 are bounded. Since x_1 and x_2 are bounded and it is given that $c(t)$ and $\dot{c}(t)$ are bounded, it follows that $\ddot{V}(t)$ is bounded. The bound on $\ddot{V}(t)$ guarantees that $\dot{V}(t)$ is uniformly continuous. In order to conclude by Barbalat's lemma we also need to prove that $\lim_{t\to\infty}\int_0^t \dot{V}(\tau)\,d\tau$ exists and is finite. This is proven according to

$$\begin{aligned} \lim_{t\to\infty}\int_0^t \dot{V}(\tau)\,d\tau &= \lim_{t\to\infty}(V(t) - V(0)) \\ &= \lim_{t\to\infty} V(t) - V(0) \end{aligned}$$

where we know that $\lim_{t\to\infty} V(t) = V_\infty$ is a finite number since $V(t) \geq 0\forall t$ and $\dot{V}(t) \leq 0\forall t$.

Solution 4

1. *The system is given by*

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2x_2 \\ \dot{x}_2 &= -x_1^3 - x_2 + u \end{aligned}$$

Let $V(x)$ be given by

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

The time derivative along the trajectories of the system is calculated as

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2\\ &= x_1\left(-x_1 + x_1^2 x_2\right) + x_2\left(-x_1^3 - x_2 + u\right)\\ &= -x_1^2 + x_1^3 x_2 - x_1^3 x_2 - x_2^2 + u x_2\\ &= -x_1^2 - x_2^2 + u x_2\\ &= -\|x\|_2^2 + u x_2\end{aligned}$$

and upper bounded as

$$\begin{aligned}\dot{V}(x) &\leq -\|x\|_2^2 + |u x_2|\\ &= -\|x\|_2^2 + |u|\,|x_2|\\ &\leq -\|x\|_2^2 + |u|\,\|x\|_2\\ &= -\|x\|_2^2 + |u|\,\|x\|_2 + \theta\|x\|_2^2 - \theta\|x\|_2^2\\ &= -(1-\theta)\|x\|_2^2 + |u|\,\|x\|_2 - \theta\|x\|_2^2\\ &= -(1-\theta)\|x\|_2^2 - (\theta\|x\|_2 - |u|)\|x\|_2\\ &\leq -(1-\theta)\|x\|_2^2 \;\; \forall\theta\|x\|_2 - |u| \geq 0\\ &= -(1-\theta)\|x\|_2^2 \;\; \forall\|x\|_2 \geq \frac{|u|}{\theta}\end{aligned}$$

where $\theta \in (0,1)$. Hence, the system is input-to-state stable.

2. *The system is given by*

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= -x_1^3 - x_2 + u\end{aligned}$$

Let $V(x)$ be given by

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

The time derivative along the trajectories of the system is calculated as

$$\begin{aligned}\dot{V}(x) &= x_1^3\dot{x}_1 + x_2\dot{x}_2\\ &= x_1^3(-x_1 + x_2) + x_2\left(-x_1^3 - x_2 + u\right)\\ &= -x_1^4 + x_1^3 x_2 - x_1^3 x_2 - x_2^2 + u x_2\\ &= -x_1^4 - x_2^2 + u x_2\end{aligned}$$

and upper bounded as

$$\begin{aligned}\dot{V}(x) &= -x_1^4 - (1-\theta)\,x_2^2 + ux_2 - \theta x_2^2 \\ &\leq -x_1^4 - (1-\theta)\,x_2^2 \;\forall\, |x_2| \geq \frac{|u|}{\theta} \end{aligned} \tag{6}$$

where $\theta \in (0,1)$. *When* $|x_2| \leq \frac{|u|}{\theta}$ *have that*

$$\begin{aligned}\dot{V}(x) &= -x_1^4 - x_2^2 + ux_2 \\ &\leq -x_1^4 - x_2^2 + |x_2|\,|u| \\ &\leq -x_1^4 - x_2^2 + \frac{|u|^2}{\theta} \\ &= -(1-\theta)\,x_1^4 - x_2^2 - \left(\theta x_1^4 - \frac{|u|^2}{\theta}\right) \\ &\leq -(1-\theta)\,x_1^4 - x_2^2 \;\forall\, |x_1| \geq \sqrt{\frac{|u|}{\theta}} \end{aligned} \tag{7}$$

By using (6) and (7) it can be recognized that

$$\dot{V}(x) \leq -(1-\theta)\left(x_1^4 + x_2^2\right) \;\; \forall\, \|x\|_\infty \geq \rho(|u|)$$

where

$$\rho(r) = \max\left(\frac{r}{\theta}, \sqrt{\frac{r}{\theta}}\right)$$

Hence, the system is input-to-state stable.

3. *With* $u = 0$ *the system is given by*

$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)\left(x_1^2 - 1\right) \\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 - 1\right)\end{aligned}$$

and it can be seen that it has an equilibrium set $\{x_1^2 = 1\}$. *Hence, the origin is not globally asymptotically stable. It follows that the system is not input-to-state stable.*

4. *The unforced system* ($u = 0$) *has equilibrium points* $(-1,-1)$, $(0,0)$ *and* $(1,1)$. *Hence, the origin is not globally asymptotically. Consequently, the system is not input-to-output stable.*

Solution 5

The system is given by $y = Hu$ *and is a series connection of*

$$\begin{aligned} y_1 &= H_1 u_1 \\ y_2 &= H_2 u_2 \end{aligned}$$

where $u = u_1$, $y_1 = u_2$ and $y = y_2$. Assume that both systems is L stable, that is

$$\begin{aligned} \|y_{1\tau}\|_L &= \|(H_1u_1)_\tau\|_L \\ &\leq \alpha_1(\|u_{1\tau}\|_L) + \beta_1 \\ \|y_{2\tau}\|_L &= \|(H_2u_2)_\tau\|_L \\ &\leq \alpha_2(\|u_{2\tau}\|_L) + \beta_2 \end{aligned}$$

The L stability of the series connection is then given by

$$\begin{aligned} \|y_\tau\|_L &= \|y_{2\tau}\|_L \\ &\leq \alpha_2(\|u_{2\tau}\|_L) + \beta_2 \\ &= \alpha_2(\|y_{1\tau}\|_L) + \beta_2 \\ &\leq \alpha_2(\alpha_1(\|u_{1\tau}\|_L) + \beta_1) + \beta_2 \\ &= \alpha_2(\alpha_1(\|u_\tau\|_L) + \beta_1) + \beta_2 \\ &\leq \alpha_2(2\alpha_1(\|u_\tau\|_L)) + \alpha_2(2\beta_1) + \beta_2 \\ &= \alpha(\|u_\tau\|_L) + \beta \end{aligned}$$

where $\alpha(\|u_\tau\|_L) = \alpha_2(2\alpha_1(\|u_\tau\|_L))$ and $\beta = \alpha_2(2\beta_1) + \beta_2$. Now assume that both systems are finite gain L stable, that is

$$\begin{aligned} \|y_{1\tau}\|_L &= \|(H_1u_1)_\tau\|_L \\ &\leq \gamma_1\|u_{1\tau}\|_L + \beta_1 \\ \|y_{2\tau}\|_L &= \|(H_2u_2)_\tau\|_L \\ &\leq \gamma_2\|u_{2\tau}\|_L + \beta_2 \end{aligned}$$

The finite gain L stability of the series connection is then given by

$$\begin{aligned} \|y_\tau\|_L &= \|y_{2\tau}\|_L \\ &\leq \gamma_2\|u_{2\tau}\|_L + \beta_2 \\ &= \gamma_2\|y_{1\tau}\|_L + \beta_2 \\ &\leq \gamma_2(\gamma_1\|u_{1\tau}\|_L + \beta_1) + \beta_2 \\ &= \gamma_1\gamma_2\|u_{1\tau}\|_L + \gamma_2\beta_1 + \beta_2 \\ &= \gamma\|u_\tau\|_L + \beta \end{aligned}$$

where $\gamma = \gamma_1\gamma_2$ and $\beta = \gamma_2\beta_1 + \beta_2$.

Solution 6

The system is given by $y = Hu$ and is a parallel connection of

$$\begin{aligned} y_1 &= H_1u_1 \\ y_2 &= H_2u_2 \end{aligned}$$

where $u = u_1 = u_2$ and $y = y_1 + y_2$. Assume that both systems is L stable, that is

$$\begin{aligned}\|y_{1\tau}\|_L &= \|(H_1u_1)_\tau\|_L \\ &\leq \alpha_1\left(\|u_{1\tau}\|_L\right) + \beta_1 \\ \|y_{2\tau}\|_L &= \|(H_2u_2)_\tau\|_L \\ &\leq \alpha_2\left(\|u_{2\tau}\|_L\right) + \beta_2\end{aligned}$$

The L stability of the parallel connection is then given by

$$\begin{aligned}\|y_\tau\|_L &= \|y_{1\tau} + y_{2\tau}\|_L \\ &\leq \|y_{1\tau}\|_L + \|y_{2\tau}\|_L \\ &\leq \alpha_1\left(\|u_{1\tau}\|_L\right) + \beta_1 + \alpha_2\left(\|u_{2\tau}\|_L\right) + \beta_2 \\ &= \alpha_1\left(\|u_\tau\|_L\right) + \beta_1 + \alpha_2\left(\|u_\tau\|_L\right) + \beta_2 \\ &= \alpha\left(\|u_\tau\|_L\right) + \beta\end{aligned}$$

where $\alpha\left(\|u_\tau\|_L\right) = \alpha_1\left(\|u_{1\tau}\|_L\right) + \alpha_2\left(\|u_{2\tau}\|_L\right)$ and $\beta = \beta_1 + \beta_2$. Now assume that both systems are finite gain L stable, that is

$$\begin{aligned}\|y_{1\tau}\|_L &= \|(H_1u_1)_\tau\|_L \\ &\leq \gamma_1\|u_{1\tau}\|_L + \beta_1 \\ \|y_{2\tau}\|_L &= \|(H_2u_2)_\tau\|_L \\ &\leq \gamma_2\|u_{2\tau}\|_L + \beta_2\end{aligned}$$

The finite gain L stability of the parallel connection is then given by

$$\begin{aligned}\|y_\tau\|_L &= \|y_{1\tau} + y_{2\tau}\|_L \\ &\leq \|y_{1\tau}\|_L + \|y_{2\tau}\|_L \\ &\leq \gamma_1\|u_{1\tau}\|_L + \beta_1 + \gamma_2\|u_{2\tau}\|_L + \beta_2 \\ &= \gamma_1\|u_\tau\|_L + \beta_1 + \gamma_2\|u_\tau\|_L + \beta_2 \\ &= \gamma\|u_\tau\|_L + \beta\end{aligned}$$

where $\gamma = \gamma_1 + \gamma_2$ and $\beta = \beta_1 + \beta_2$.

Solution 7
We have that

$$\begin{aligned}y_{1\tau} &= (H_1e_1)_\tau \\ y_{2\tau} &= (H_2e_2)_\tau\end{aligned}$$

and that

$$\begin{aligned}\|y_{1\tau}\|_L &= \|(H_1e_1)_\tau\|_L \\ &\leq \gamma_1\|e_{1\tau}\|_L+\beta_1 \\ \|y_{2\tau}\|_L &= \|(H_2e_2)_\tau\|_L \\ &\leq \gamma_2\|e_{2\tau}\|_L+\beta_2\end{aligned}$$

Evaluating $\|y_{1\tau}\|_L$ *gives*

$$\begin{aligned}\|y_{1\tau}\|_L &\leq \gamma_1\|e_{1\tau}\|_L+\beta_1 \\ &= \gamma_1\|u_{1\tau}-y_{2\tau}\|_L+\beta_1 \\ &\leq \gamma_1\|u_{1\tau}\|_L+\gamma_1\|y_{2\tau}\|_L+\beta_1 \\ &\leq \gamma_1\|u_{1\tau}\|+\gamma_1(\gamma_2\|e_{2\tau}\|_L+\beta_2)+\beta_1 \\ &= \gamma_1\|u_{1\tau}\|+\gamma_1\gamma_2\|e_{2\tau}\|_L+\gamma_1\beta_2+\beta_1 \\ &= \gamma_1\|u_{1\tau}\|+\gamma_1\gamma_2\|u_{2\tau}+y_{1\tau}\|_L+\gamma_1\beta_2+\beta_1 \\ &\leq \gamma_1\|u_{1\tau}\|+\gamma_1\gamma_2\|u_{2\tau}\|_L+\gamma_1\gamma_2\|y_{1\tau}\|_L+\gamma_1\beta_2+\beta_1 \\ &\Rightarrow \|y_{1\tau}\|_L \leq \frac{1}{1-\gamma_1\gamma_2}(\gamma_1\|u_{1\tau}\|_L+\gamma_1\gamma_2\|u_{2\tau}\|_L+\gamma_1\beta_2+\beta_1)\end{aligned} \quad (8)$$

and evaluating $\|y_{2\tau}\|_L$ *gives*

$$\begin{aligned}\|y_{2\tau}\|_L &\leq \gamma_2\|e_{2\tau}\|_L+\beta_2 \\ &= \gamma_2\|u_{2\tau}+y_{1\tau}\|_L+\beta_2 \\ &\leq \gamma_2\|u_{2\tau}\|_L+\gamma_2\|y_{1\tau}\|_L+\beta_2 \\ &\leq \gamma_2\|u_{2\tau}\|_L+\gamma_2(\gamma_1\|e_{1\tau}\|_L+\beta_1)+\beta_2 \\ &= \gamma_2\|u_{2\tau}\|_L+\gamma_2\gamma_1\|e_{1\tau}\|_L+\gamma_2+\beta_1+\beta_2 \\ &= \gamma_2\|u_{2\tau}\|_L+\gamma_2\gamma_1\|u_{1\tau}-y_{2\tau}\|_L+\gamma_2+\beta_1+\beta_2 \\ &\leq \gamma_2\|u_{2\tau}\|_L+\gamma_2\gamma_1\|u_{1\tau}\|_L+\gamma_2\gamma_1\|y_{2\tau}\|_L+\gamma_2+\beta_1+\beta_2 \\ &\Rightarrow \|y_{2\tau}\|_L \leq \frac{1}{1-\gamma_1\gamma_2}(\gamma_2\|u_{2\tau}\|_L+\gamma_2\gamma_1\|u_{1\tau}\|_L+\gamma_2+\beta_1+\beta_2)\end{aligned} \quad (9)$$

From (8) and (9) it can be seen that the map from (u_1,u_2) *to* (y_1,y_2) *is finite-gain L stable if and only is the map from* (e_1,e_2) *to* (y_1,y_2) *is finite-gain L stable. This can be seen from*

$$\begin{aligned}\|y_{1\tau}\|_L &\leq \frac{1}{1-\gamma_1\gamma_2}(\gamma_1\|u_{1\tau}\|+\gamma_1\gamma_2\|u_{2\tau}\|_L+\gamma_1\beta_2+\beta_1) \\ \|e_{1\tau}\|_L &\leq \left(\gamma_1+\frac{\gamma_1^2}{(1-\gamma_1\gamma_2)}\right)\|u_{1\tau}\|_L+\left(1+\frac{\gamma_1^2\gamma_2}{(1-\gamma_1\gamma_2)}\right)\|u_{2\tau}\|_L \\ &\quad +\frac{\gamma_1(\gamma_1\beta_2+\beta_1)}{(1-\gamma_1\gamma_2)}(\gamma_1\beta_2+\beta_1)+\gamma_2^2+\gamma_2\beta_1\end{aligned}$$

where it has been used that $\gamma_1 \|e_{1\tau}\|_L + \beta_1 \le \gamma_1 \|u_{1\tau}\| + \gamma_1\gamma_2 \|u_{2\tau}\|_L + \gamma_1\gamma_2 \|y_{1\tau}\|_L + \gamma_1\beta_2 + \beta_1$ *and (7). Similar results can be found for* $\|y_{2\tau}\|_L$ *and* $\|e_{2\tau}\|_L$.

Solution 8

Let the input to the system be denoted $\tilde{u}$ *and the output be denoted* $\tilde{y}$. *From the block diagram we have the following relations*

$$\begin{aligned} \tilde{y} &= h(t,u) - K_1 u \\ \tilde{u} + \tilde{y} &= Ku \end{aligned}$$

From the sector condition we have that

$$\begin{gathered} (h(t,u) - K_1 u)^T (h(t,u) - K_2 u) \le 0 \\ K = K_2 - K_1 = K^T > 0 \end{gathered} \tag{10}$$

Evaluating the block diagram it can be seen that

$$h(t,u) - K_1 u = \tilde{y} \tag{11}$$

and that

$$\begin{aligned} h(t,u) - K_2 u &= h(t,u) - K_2 u - K_1 u + K_1 u \\ &= \tilde{y} - (K_2 - K_1) u \\ &= \tilde{y} - Ku \\ &= \tilde{y} - \tilde{u} - \tilde{y} \\ &= -\tilde{u} \end{aligned} \tag{12}$$

Using (11), (12) the sector condition (10) it can be recognized that

$$\begin{aligned} (h(t,u) - K_1 u)^T (h(t,u) - K_2 u) &= \tilde{y}^T(-\tilde{u}) \\ &= -\tilde{u}^T \tilde{y} \\ &\le 0 \\ &\Rightarrow \tilde{u}^T \tilde{y} \ge 0 \end{aligned}$$

which implies that the system is passive from $\tilde{u}$ *to* $\tilde{y}$.

Solution 9

The system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - a x_2 + u \\ y &= k x_2 + u \end{aligned}$$

where

$$\begin{aligned}
a &> 0 \\
k &> 0 \\
h &\in [\alpha_1, \infty] \\
&\Rightarrow zh(z) \geq \alpha_1 z^2 \\
\alpha_1 &> 0
\end{aligned}$$

A storage function is given by

$$V(x) = k \int_0^{x_1} h(z)\, dz + x^T P x$$

where $p_{11} = ap_{12}$, $p_{22} = \frac{k}{2}$ *and* $0 < p_{12} < \min\left\{2\alpha_1,\ \frac{ak}{2}\right\}$. *The time derivative of the storage functions along the trajectories of the system is found as*

$$\begin{aligned}
\dot{V}(x) &= k\frac{\partial}{\partial x_1}\left(\int_0^{x_1} h(z)\, dz\right)\dot{x}_1 + \dot{x}^T P x + x^T P \dot{x} \\
&= kh(x_1)\dot{x}_1 + 2x^T P \dot{x} \\
&= kh(x_1)x_2 - h(x_1)kx_2 + kux_2 - 2h(x_1)x_1 p_{12} + 2ux_1 p_{12} - akx_2^2 + 2x_2^2 p_{12} \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 + kux_2 \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 + (kx_2 + u)u - u^2 \\
&= -akx_2^2 + 2p_{12}x_2^2 - 2p_{12}h(x_1)x_1 + 2p_{12}ux_1 - u^2 + yu
\end{aligned}$$

By rewriting this last expression it can be seen that

$$\begin{aligned}
yu &= \dot{V}(x) + akx_2^2 - 2p_{12}x_2^2 + 2p_{12}h(x_1)x_1 - 2p_{12}ux_1 + u^2 \\
&= \dot{V}(x) + (ak - 2p_{12})x_2^2 + 2p_{12}h(x_1)x_1 + (u - p_{12}x_1)^2 - p_{12}^2 x_1^2 \\
&\geq \dot{V}(x) + (ak - 2p_{12})x_2^2 + 2p_{12}\alpha_1 x - p_{12}^2 x_1^2 + (u - p_{12}x_1)^2 \\
&= \dot{V}(x) + (ak - 2p_{12})x_2^2 + \left(2p_{12}\alpha_1 - p_{12}^2\right)x_1^2 + (u - p_{12}x_1)^2 \\
&\geq \dot{V}(x) + (ak - 2p_{12})x_2^2 + \left(2p_{12}\alpha_1 - p_{12}^2\right)x_1^2 \\
&= \dot{V}(x) + \psi(x)
\end{aligned}$$

where

$$\psi(x) = (ak - 2p_{12})x_2^2 + p_{12}(2\alpha_1 - p_{12})x_1^2$$

Since $0 < p_{12} < \min\left\{2\alpha_1,\ \frac{ak}{2}\right\}$ *we have that* $\psi(x)$ *is positive definite. Hence, the system is strictly passive.*

Solution 10

A parallel connection is characterized by

$$\begin{aligned} u &= u_1 = u_2 \\ y &= y_1 + y_2 \end{aligned}$$

where the two systems is given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u_1) \\ \dot{x}_2 &= f(x_2, u) \end{aligned}$$

with the storage functions $V_1(x_1)$ *and* $V_2(x_2)$. *The various passivity properties of the system may be expressed as*

$$\begin{aligned} u_1^T y_1 &\geq \frac{\partial V_1(x_1)}{x_1} f_1(x_1, u_1) + u_1^T \varphi_1(u_1) + y_1^T \rho_1(y_1) + \psi_1(x_1) \\ u_2^T y_2 &\geq \frac{\partial V_2(x_2)}{x_2} f_2(x_2, u_2) + u_2^T \varphi_2(u_2) + y_2^T \rho_2(y_2) + \psi_2(x_2) \end{aligned}$$

Using the structure of the parallel connection it can be recognized that

$$\begin{aligned} u^T y &= u^T (y_1 + y_2) \\ &= u_1^T y_1 + u_2^T y_1 \\ &\geq \frac{\partial V_1(x_1)}{x_1} f_1(x_1, u_1) + u_1^T \varphi_1(u_1) + y_1^T \rho_1(y_1) + \psi_1(x_1) \\ &\quad + \frac{\partial V_2(x_2)}{x_2} f_2(x_2, u_2) + u_2^T \varphi_2(u_2) + y_2^T \rho_2(y_2) + \psi_2(x_2) \\ &= \begin{bmatrix} \frac{\partial V_1(x_1)}{x_1} & \frac{\partial V_2(x_2)}{x_2} \end{bmatrix} \begin{bmatrix} f_1(x_1, u_1) \\ f_2(x_2, u_2) \end{bmatrix} + u^T \varphi_1(u) + u^T \varphi_2(u) \\ &\quad + y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) + \psi_1(x_1) + \psi_2(x_2) \\ &= \frac{\partial}{\partial x} V(x) f(x, u) + u^T \varphi(u) + y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) + \psi(x) \\ &= \dot{V}(x) + u^T \varphi(u) + y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) + \psi(x) \end{aligned} \tag{13}$$

where

$$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \tag{14}$$

$$V(x) = V_1(x_1) + V_2(x_2) \tag{15}$$

$$\varphi(u) = \varphi_1(u) + \varphi_2(u) \tag{16}$$

$$\psi(x) = \psi_1(x_1) + \psi_2(x_2) \tag{17}$$

From (13) and (14)-(17) it can be seen that the parallel connection keeps the passivity properties of passive, input srictly passive and strictly passive. For output strictly passive, we require that

$$y_i^T \rho_i(y_i) \geq \delta_i y_i^T y_i \tag{18}$$

for some positive δ_i. *Using (18) and* $\delta = \max\{\delta_1, \delta_2\}$ *we may rewrite* $y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2)$ *according to*

$$\begin{aligned} y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) &\geq \delta_1 y_1^T y_1 + \delta_2 y_2^T y_2 \\ &\geq \delta y_1^T y_1 + \delta y_2^T y_2 \\ &= \delta\left(y_1^T y_1 + y_2^T y_2\right) \\ &\geq \delta\left(\frac{1}{2}(y_1+y_2)^T(y_1+y_2)\right) \\ &= \frac{1}{2}\delta y^T y \end{aligned}$$

where it has been used that

$$(y_1+y_2)^T(y_1+y_2) \leq 2\left(y_1^T y_1 + y_2^T y_2\right)$$

Solution 11
The system is given by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

where

- $M(q) = M^T(q) > 0 \ \forall q \in \mathbb{R}^m$
- $\dot{M}(q) - 2C(q,\dot{q})$ *is skew-symetric* $\forall q, \dot{q} \in \mathbb{R}^m$
- $D = D^T > 0$
- $P(q) > 0$ *is the total potential energy of the links due to gravity (a positive definite function)*
- $g(q) = \left[\frac{\partial P(q)}{\partial q}\right]^T$ *where* $g(q)$ *has an isolated root at* $q = 0$.

1. *The storage function is taken as*

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$$

The time derivative along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}(q,\dot{q}) &= \frac{1}{2}\ddot{q}^T M(q)\dot{q} + \frac{1}{2}\dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \dot{P}(q) \\
&= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{\partial}{\partial q}P(q)\dot{q} \\
&= \dot{q}^T\left(u - C(q,\dot{q})\dot{q} - D\dot{q} - g(q)\right) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + g^T(q)\dot{q} \\
&= \dot{q}^T u - \dot{q}^T C(q,\dot{q})\dot{q} - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + g^T(q)\dot{q} \\
&= \dot{q}^T u - \dot{q}^T D\dot{q} + \dot{q}^T\left(\frac{1}{2}\dot{M}(q) - C(q,\dot{q})\right)\dot{q} \\
&= \dot{q}^T u - \dot{q}^T D\dot{q} \\
&\leq \dot{q}^T u \\
&\Rightarrow u^T\dot{q} \geq \dot{V}(q,\dot{q})
\end{aligned}$$

Hence, the map from u to $\dot{q}$ is passive.

2. *Using the control input*

$$u = -K_d\dot{q} + v$$

where K_d is a diagonal positive definite matrix, the time derivative of the storage function along the trajectories of the system is found as

$$\begin{aligned}
\dot{V}(q,\dot{q}) &= \dot{q}^T u - \dot{q}^T D\dot{q} \\
&= \dot{q}^T\left(-K_d\dot{q} + v\right) - \dot{q}^T D\dot{q} \\
&= -\dot{q}^T K_d\dot{q} + \dot{q}^T v - \dot{q}^T D\dot{q} \\
&\leq -\lambda_{\min}(K_d)\|\dot{q}\|_2^2 + \dot{q}^T v \\
&\Rightarrow u^T\dot{q} \geq \dot{V}(q,\dot{q}) + \lambda_{\min}(K_d)\dot{q}^T\dot{q}
\end{aligned}$$

Hence, the map from u to $\dot{q}$ is output strictly passive.

3. *The storage function is a positive definite function in q and $\dot{q}$. Its time derivative is negative semidefinite. By applying LaSalle's theorem, it can be recognized that the origin is asymptotically stable. It will be globally asymptotically stable if $q = 0$ is the unique root of $g(q) = 0$ and $P(q)$ is radially unbounded.*