TTK4150 Nonlinear Control Systems
Solution 3

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Solution 1

The function is given by

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

1. *Let* $x_1 = 0$, *then* $V(x)$ *is given by*

$$V(x) = \frac{x_2^2}{1 + x_2^2} + x_2^2$$

and it can be seen that $V(x) = \frac{x_2^2}{1+x_2^2} + x_2^2 \to \infty$ *as* $|x_2| \to \infty$. *Let* $x_2 = 0$, *then* $V(x)$ *is given by*

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_1^2$$

and it can be seen that $V(x) = \frac{x_1^2}{1+x_1^2} + x_1^2 \to \infty$ *as* $|x_1| \to \infty$.

2. *On the set* $x_1 = x_2$ *the function is given by*

$$V(x) = \frac{4x_1^2}{1 + 4x_1^2}$$

and it can be seen that $V(x) = \frac{4x_1^2}{1+4x_1^2} \to 1$ *as* $|x_1| \to \infty$.

Solution 2

1. *Given* $f(x) = \int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, x d\sigma$

$$\begin{aligned}
x^T P f(x) + f^T(x) P x &= x^T P \int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, x d\sigma + \left(\int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, x d\sigma \right)^T P x \\
&= x^T P \int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, x d\sigma + \int_0^1 x^T \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T d\sigma P x \\
&= x^T \left(P \int_0^1 \frac{\partial}{\partial x} f(\sigma x)\, d\sigma + \int_0^1 \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T d\sigma P \right) x \\
&= x^T \int_0^1 \left(P \frac{\partial}{\partial x} f(\sigma x) + \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T P \right) d\sigma x
\end{aligned}$$

and by using $P \frac{\partial}{\partial x} f(\sigma x) + \left(\frac{\partial}{\partial x} f(\sigma x) \right)^T P \leq -I$ *the expression may be upper bounded by*

$$x^T P f(x) + f^T(x) P x \leq x^T (-I) x = -x^T x = -\|x\|_2^2$$

2. *Given the function* $V(x) = f^T(x) P f(x)$ *where* P *is symmetric and positive definite. To show that* $V(x)$ *is positive definite, we need to show that* $f(x) = 0$ *if and only if* $x = 0$*. In other words we need to show that the origin is a unique equilibrium point. Suppose, to the contrary that there is a* $p \neq 0$ *such that* $f(p) = 0$*. Then*

$$p^T p \leq -\left(p^T P f(p) + f^T(p) P p \right) = 0$$

which is a contradiction since $p \neq 0$ *(in order to satisfy the above inequality* p *needs to equal zero). Hence the origin is a unique equilibrium point. To see that the function is radially unbounded notice that*

$$\begin{aligned}
\frac{x^T P f(x)}{\|x\|_2^2} &= \frac{x^T P f(x)}{2\|x\|_2^2} + \frac{f^T(x) P x}{2\|x\|_2^2} \\
&= \frac{1}{2\|x\|_2^2} \left(x^T P f(x) + f^T(x) P x \right) \\
&\leq \frac{1}{2\|x\|_2^2} \left(-\|x\|_2^2 \right) \\
&= -\frac{1}{2}
\end{aligned}$$

Suppose now that $\|f(x)\|_2 \le c$ *as* $\|x\|_2 \to \infty$. *Then*

$$\begin{aligned}
\frac{\left\|x^T P f(x)\right\|_2}{\|x\|_2^2} &\le \frac{\left\|x^T\right\|_2 \|P\|_2 \|f(x)\|_2}{\|x\|_2^2} \\
&\le \frac{\left\|x^T\right\|_2 \|P\|_2 c}{\|x\|_2^2} \\
&= \frac{\|P\|_2 c}{\|x\|_2}
\end{aligned}$$

tends to zero when $\|x\|_2 \to 0$ *which is a contradiction to*

$$\frac{x^T P f(x)}{\|x\|_2^2} \le -\frac{1}{2}$$

It follows that the function $V(x)$ *is radially unbounded* ($\|f(x)\|_2 \to \infty$ *as* $\|x\|_2 \to \infty$).

3. *We have sown that* $V(x)$ *is positive definite and radially unbounded. The time derivative of the function is found as*

$$\begin{aligned}
\dot{V}(x) &= \dot{f}^T(x) P f(x) + f^T(x) P \dot{f}(x) \\
&= \left(\frac{\partial}{\partial x} f(x)\dot{x}\right)^T P f(x) + f^T(x) P \left(\frac{\partial}{\partial x} f(x)\dot{x}\right) \\
&= \left(\frac{\partial f(x)}{\partial x} f(x)\right)^T P f(x) + f^T(x) P \left(\frac{\partial f(x)}{\partial x} f(x)\right) \\
&= f^T(x)\left(\frac{\partial f(x)}{\partial x}\right)^T P f(x) + f^T(x) P \left(\frac{\partial f(x)}{\partial x} f(x)\right) \\
&= f^T(x)\left(P\frac{\partial f(x)}{\partial x} + \left(\frac{\partial f(x)}{\partial x}\right)^T P\right) f(x) \\
&\le -f^T(x) f(x) \\
&= -\|f(x)\|_2^2
\end{aligned}$$

Since origin is a unique equilibrium point and all of the conditions are globally, the origin is a globally asymptotically stable equilibrium point.

Solution 3

The system is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g(x_1)(x_1 + x_2)
\end{aligned}$$

where it can be seen that the origin is a unique equilibrium point. Using $g(y) \geq 1 \; \forall y$ it can be recognized that

$$\begin{aligned} \int_0^{x_1} y g(y)\, dy &\geq \int_0^{x_1} y dy \\ &= \frac{1}{2} x_1^2 \end{aligned}$$

Using this, the function $V(x)$ is lover bounded by

$$\begin{aligned} V(x) &= \int_0^{x_1} y g(y)\, dy + x_1 x_2 + x_2^2 \\ &\geq \frac{1}{2} x_1^2 + x_1 x_2 + x_2^2 \\ &= \frac{1}{2} x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x \end{aligned}$$

which shows that the function is positive definite and radially unbounded. The time derivative of the function is found as

$$\begin{aligned} \dot{V}(x) &= (x_1 g(x_1) + x_2)\dot{x}_1 + (x_1 + 2x_2)\dot{x}_2 \\ &= (x_1 g(x_1) + x_2) x_2 + (x_1 + 2x_2)(-g(x_1)(x_1 + x_2)) \\ &= g(x_1) x_1 x_2 + x_2^2 - g(x_1) x_1^2 - g(x_1) x_1 x_2 - 2g(x_1) x_1 x_2 - g(x_1) 2x_2^2 \\ &= x_2^2 - g(x_1) x_1^2 - 2g(x_1) x_1 x_2 - g(x_1) 2x_2^2 \\ &= -g(x_1)\left(x_1^2 + 2x_1 x_2 + 2x_2^2\right) + x_2^2 \\ &= -g(x_1) x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + x_2^2 \\ &= -g(x_1) x^T Q x + x_2^2 \end{aligned}$$

Since Q is positive definite and $g(x_1) \geq 1$, the time derivative may be upper bounded by

$$\begin{aligned} \dot{V}(x) &\leq -x^T Q x + x_2^2 \\ &= -\left(x_1^2 + 2x_1 x_2 + 2x_2^2\right) + x_2^2 \\ &= -\left(x_1^2 + 2x_1 x_2 + x_2^2\right) \\ &= -(x_1 + x_2)^2 \end{aligned}$$

and it follows that $\dot{V}(x)$ is negative semi definite. Using Corollary 4.2 it can be recognized that the set s is given by

$$S = \left\{ x \in \mathbb{R}^2 \middle| \, x_1 = -x_2 \right\}$$

and it can be seen from the system equation that no solution can stay identical in S other than the trivial solution $x = 0$, and globally asymptotically stability of the origin follows.

Solution 4

The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - x_2 - h_2(h_3) \\ \dot{x}_3 &= x_2 - x_3\end{aligned}$$

1. *From the system equations it can be seen that the equilibrium point is given by*

$$\begin{aligned}0 &= x_2 \\ 0 &= -h_1(x_1) - h_2(x_3) \\ 0 &= x_2 - x_3\end{aligned}$$

   which is equivalent to

$$\begin{aligned}x_2 &= 0 \\ -h_1(x_1) - h_2(0) &= 0 \\ x_3 &= 0\end{aligned}$$

   since $h_3(0) = 0$ and $h_1(x_1) = 0$ only when $x_1 = 0$, origin is a unique equilibrium point.

2. *Since $V(x)$ is a sum of nonnegative functions functions ($h_i(y) \geq 0\ \forall y \geq 0$) it is a positive semi definite function. To show that it is positive definite, we need to show that*

$$V(x) = 0 \Rightarrow x = 0$$

   Since $yh_i(y) > 0\ \forall y \neq 0$, the integral $\int_0^z h_i(y)\,dy$ vanish if and only if $x_i = 0$, and it follows that $V(x)$ is positive definite.

3. *The time derivative of the function*

$$V(x) = \int_0^{x_1} h_1(y)\,dy + \frac{1}{2}x_2^2 + \int_0^{x_3} h_2(y)\,dy$$

   along the trajectories of the system is found as

$$\begin{aligned}\dot{V}(x) &= h_1(x_1)\dot{x}_1 + x_2\dot{x}_2 + h_2(x_3)\dot{x}_3 \\ &= h_1(x_1)x_2 + x_2(-h_1(x_1) - x_2 - h_2(x_3)) + h_2(x_3)(x_2 - x_3) \\ &= -x_2^2 - h_2(x_3)x_3 \\ &= -\left(x_2^2 + h_2(x_3)x_3\right)\end{aligned}$$

since $h_2(x_3)\,x_3 > 0\;\forall x_3 \neq 0$ *we have that* $\dot{V}(x)$ *is negative semi definite. In order to prove asymptotic stability, we apply Corollary 4.1. From* $\dot{V}(x)$ *it can be seen that the set* S *is given by*

$$S = \left\{ x \in \mathbb{R}^3 \middle| \, x_2 + x_3 = 0 \right\}$$

and it can be seen from the system equation that no solution can stay identical in S *other than the trivial solution* $x = 0$*, and asymptotic stability of the origin follows.*

4. *To show global asymptotically stability the function* $V(x)$ *need to be radially unbounded. This is the case if the functions* h_i *satisfies* $\int_0^z h_i(y)\,dy \to \infty$ *as* $|z| \to \infty$*.*

Solution 5
If $r_1 \geq r_2$ *we have that* $r_1 + r_2 \leq 2r_1$ *which implies that*

$$\alpha(r_1 + r_2) \leq \alpha(2r_1) \leq \alpha(2r_1) + \alpha(2r_2)$$

and if $r_2 \geq r_1$ *we have that* $r_1 + r_2 \leq 2r_2$ *which implies that*

$$\alpha(r_1 + r_2) \leq \alpha(2r_2) \leq \alpha(2r_1) + \alpha(2r_2)$$

where it has been used that a class K *function is strictly increasing in its argument. Using the two different cases, we can conclude that the inequality* $\alpha(r_1 + r_2) \leq \alpha(2r_1) + \alpha(2r_2)$ *is always satisfied.*

Solution 6
The system is given by

$$\begin{aligned} \dot{x}_1 &= \frac{1}{L(t)} x_2 \\ \dot{x}_2 &= -\frac{1}{C(t)} x_1 - \frac{R(t)}{L(t)} x_2 \end{aligned}$$

where $L(t)$, $C(t)$ *and* $R(t)$ *continuously differentiable and bounded from below and above. The Lyapunov function candidate is given by*

$$V(t,x) = \left(R(t) + \frac{2L(t)}{R(t)\,C(t)} \right) x_1^2 + 2x_1x_2 + \frac{2}{R(t)} x_2^2$$

1. *The function can be upper bounded by*

$$V(t,x) \leq \left(k_6 + \frac{2k_2}{k_3k_5} \right) x_1^2 + 2x_1x_2 + \frac{2}{k_5} x_2^2$$

and lower bounded by

$$V(t,x) \geq \left(k_5 + \frac{2k_1}{k_4 k_4}\right) x_1^2 + 2x_1 x_2 + \frac{2}{k_6} x_2^2$$

Using the upper bounds it is clear that $V(t,x)$ is decresent. If we try to use the lower bounds to show that $V(t,x)$ is positive definite, we will have to restrict the constants to

$$\frac{2k_5}{k_6} + \frac{4k_1}{k_6^2 k_4} - 1 > 0$$

Instead of making this restriction, we work directly with $V(t,x)$ and rewrite it as

$$\begin{aligned} V(t,x) &= x^T \begin{bmatrix} \left(R(t) + \frac{2L(t)}{R(t)C(t)}\right) & 1 \\ 1 & \frac{2}{R(t)} \end{bmatrix} x \\ &\geq x^T \begin{bmatrix} R(t) & 1 \\ 1 & \frac{2}{R(t)} \end{bmatrix} x \\ &= x^T \tilde{P} x \end{aligned}$$

The eigenvalues of $\tilde{P}$ are calculated as

$$\lambda_{1,2} = \begin{cases} \frac{1}{R}\left(\frac{1}{2}R^2 - \frac{1}{2}\sqrt{R^4+4} + 1\right) \\ \frac{1}{R}\left(\frac{1}{2}R^2 + \frac{1}{2}\sqrt{R^4+4} + 1\right) \end{cases}$$

The smallest eigenvalue is given by

$$\begin{aligned} \lambda_{\min} &= \frac{1}{2}\left(\left(R + \frac{2}{R}\right) - \sqrt{R^2 + \frac{4}{R^2}}\right) \\ &= \frac{1}{2}\left(\left(R + \frac{2}{R}\right) - \sqrt{\left(R + \frac{2}{R}\right)^2 - 4}\right) \end{aligned}$$

where it is easily seen that there are positive constants c_1 and c_2 such that $\left(R + \frac{2}{R}\right)^2 - 4 \geq c_1$ and $\lambda_{\min} \geq c_2$ for all t, which shows that $V(t,x)$ is positive definite.

2. *The time derivative of $V(t,x)$ is found as*

$$\begin{aligned} \dot{V}(t,x) &= -\frac{2}{C(t)}\left(1 + \dot{R}(t)\left(\frac{L(t)}{R^2(t)} - \frac{C(t)}{2}\right) + \frac{L(t)\dot{C}(t)}{R(t)C(t)} - \frac{\dot{L}(t)}{R(t)}\right) x_1^2 \\ &\quad -\frac{2}{L(t)}\left(1 + \frac{L(t)\dot{R}(t)}{R^2(t)}\right) x_2^2 \end{aligned}$$

Suppose $\dot{L}(t)$, $\dot{C}(t)$ and $\dot{R}(t)$ satisfy

$$\begin{aligned} 1+\dot{R}(t)\left(\frac{L(t)}{R^2(t)}-\frac{C(t)}{2}\right)+\frac{L(t)\dot{C}(t)}{R(t)C(t)}-\frac{\dot{L}(t)}{R(t)} &> c_3 \\ 1+\frac{L(t)\dot{R}(t)}{R^2(t)} &> c_4 \end{aligned}$$

Then

$$\dot{V}(t,x) < -\frac{2c_3}{k_3}x_1^2 - \frac{2c_4}{k_1}$$

and $\dot{V}(t,x)$ is negative definite. This implies that the origin is uniformly asymptotically stable. Using Theorem 4.10 it is concluded that the origin is exponentially stable.

Solution 7

The system is given by

$$\begin{aligned} \dot{x}_1 &= h(t)x_2 - g(t)x_1^3 \\ \dot{x}_2 &= -h(t)x_1 - g(t)x_2^3 \end{aligned}$$

where $h(t)$ and $g(t)$ are bounded, continuously differentiable functions and $g(t) \geq k > 0 \ \forall t \geq 0$.

1. *It can be recognized from the model that $x=0$ is a equilibrium point. The stability properties are analyzed using the Lyapunov function candidate*

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

The time derivative along the trajectories of the system is found as

$$\begin{aligned} \dot{V}(x) &= x_1\left(h(t)x_2 - g(t)x_1^3\right) + x_2\left(-h(t)x_1 - g(t)x_2^3\right) \\ &= -g(t)x_1^4 + h(t)x_1x_2 - h(t)x_1x_2 - g(t)x_2^4 \\ &= -g(t)x_1^4 - g(t)x_2^4 \\ &= -g(t)\left(x_1^4 + x_2^4\right) \\ &\leq -k\left(x_1^4 + x_2^4\right) \end{aligned}$$

Hence, the origin is uniformly asymptotically stable.

2. *The Lyapunov function does not satisfy Theorem 4.10. The next step is to use Theorem 4.15, where*

$$\begin{aligned} A(t) &= \left.\frac{\partial f(t,x)}{\partial x}\right|_{x=0} \\ &= \begin{bmatrix} 0 & -h(t) \\ -h(t) & 0 \end{bmatrix} \end{aligned}$$

Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

The time derivative along the trajectories of the system is found as

$$\begin{aligned}\dot{V}(x) &= x_1 h(t) x_2 - x_2 h(t) x_1 \\ &= 0\end{aligned}$$

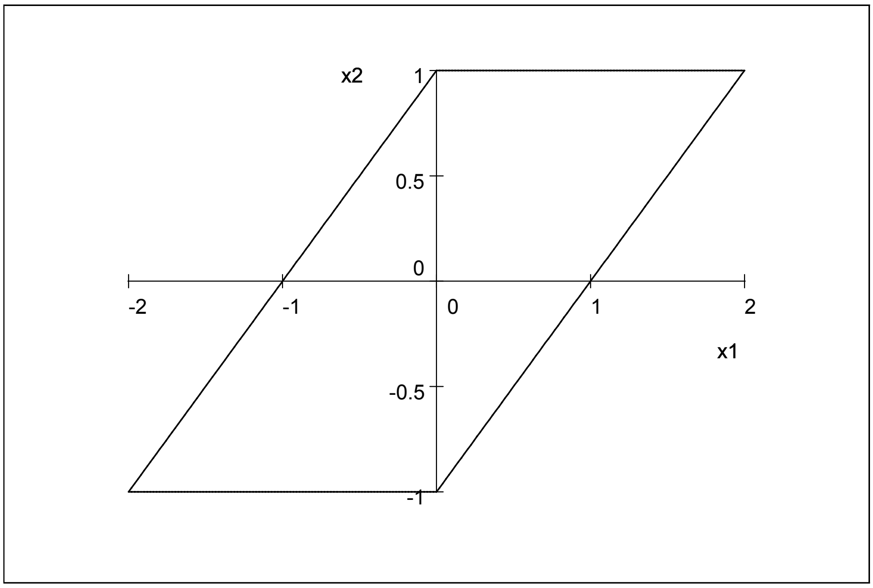
This shows that a solution starting at $V(x) = c$ remains on that set $\frac{1}{2}(x_1^2 + x_2^2) = c$ for all t, by which we conclude that the origin of the linear system $\dot{x} = A(t)x$ is not exponentially stable. Moreover, using Theorem 4.15 we conclude that the origin of the system $\dot{x} = f(t,x)$ is not exponentially stable.

3. *Since $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is a radially unbounded Lyapunov function for the system with a time derivative satisfying $\dot{V}(x) \leq -k(x_1^4 + x_2^4)$ globally, we conclude by Theorem 4.9 that the origin is globally uniformly asymptotically stable.*

4. *Since the system is not exponentially stable, it can not be globally exponentially stable.*

Solution 8

The set D in the phase plane is found as



where

$$\partial D = \left\{ \begin{array}{c} x_2 = -1 | -2 \leq x_1 \leq 0 \\ x_2 = 1 | \, 0 \leq x_1 \leq 2 \\ x_2 = x_1 - 1 | -2 \leq x_1 \leq 0 \\ x_2 = x_1 + 1 | \, 0 \leq x_1 \leq 2 \end{array} \right\}$$

To estimate the region of attraction, we calculate

$$c = \min_{x \in \partial D} V(x)$$

and the estimate is then given by the set

$$\left\{ x \in \mathbb{R}^2 \middle| \, V(x) < c \right\}$$

since this set will be contained in D *and all trajectories starting in this set will remain in this set and since* $\dot{V}(x) < 0$. *Using* ∂D *the following is found*

$$\begin{aligned} \min_{x_2=-1|-2\leq x_1\leq 0} V(x) &= \min_{x_2=-1|-2\leq x_1\leq 0} \left(x_1^2 + x_2^2\right) \\ &= \min_{x_2=-1|-2\leq x_1\leq 0} \left(\left(1 + x_2^2\right)\right) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \min_{x_2=1|0\leq x_1\leq 2} V(x) &= \min_{x_2=1|0\leq x_1\leq 2} \left(x_1^2 + x_2^2\right) \\ &= \min_{x_2=1|0\leq x_1\leq 2} \left(x_1^2 + 1\right) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \min_{x_2=x_1-1|-2\leq x_1\leq 0} V(x) &= \min_{x_2=x_1-1|-2\leq x_1\leq 0} \left(x_1^2 + x_2^2\right) \\ &= \min_{-2\leq x_1\leq 0} \left(x_1^2 + (x_1-1)^2\right) \\ &= \min_{-2\leq x_1\leq 0} \left(2x_1^2 - 2x_1 + 1\right) \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \min_{x_2=x_1+1|0\leq x_1\leq 2} V(x) &= \min_{x_2=x_1+1|0\leq x_1\leq 2} \left(x_1^2 + x_2^2\right) \\ &= \min_{0\leq x_1\leq 2} \left(x_1^2 + (x_1+1)^2\right) \\ &= \min_{0\leq x_1\leq 2} \left(2x_1^2 + 2x_1 + 1\right) \\ &= \frac{1}{2} \end{aligned}$$

which gives $c = \frac{1}{2}$*. A estimate of the region of attraction is then given by* $\left\{ x \in \mathbb{R}^2 |\, x_1^2 + x_2^2 < \frac{1}{2} \right\}$.