TTK4150 Nonlinear Control Systems
Solution 2

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Fall 2003

Solution 1

1. *The Jacobian matrix evaluated at $x = 0$ is given by*

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{x=0} \\ &= \left.\begin{bmatrix} -1 & 2x_2 \\ 0 & -1 \end{bmatrix}\right|_{x=0} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

and the eigenvalues are calculated as

$$\lambda_{1,2} = -1$$

Using Lyapunov's indirect method, it is concluded that the origin is asymptotically stable. Using phase plane analysis, it is concluded that the origin is a stable node.

2. *The Jacobian matrix evaluated at $x = 0$ is given by*

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{x=0} \\ &= \left.\begin{bmatrix} 3x_1^2 - 2x_1x_2 + x_2^2 - 1 & 2x_1x_2 - x_1^2 - 3x_2^2 + 1 \\ 2x_1x_2 + 3x_1^2 + x_2^2 - 1 & 2x_1x_2 + x_1^2 + 3x_2^2 - 1 \end{bmatrix}\right|_{x=0} \\ &= \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \end{aligned}$$

and the eigenvalues are calculated as

$$\lambda_{1,2} = -1 \pm i$$

Using Lyapunov's indirect method, it is concluded that the origin is asymptotically stable. Using phase plane analysis, it is concluded that the origin is a stable focus.

3. *The Jacobian matrix evaluated at $x = 0$ is given by*

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{x=0} \\ &= \left.\begin{bmatrix} -1 & -1 \\ 1 & 3x_2^2 \end{bmatrix}\right|_{x=0} \\ &= \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

and the eigenvalues are calculated as

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Using Lyapunov's indirect method, it is concluded that the origin is asymptotically stable. Using phase plane analysis, it is concluded that the origin is a stable focus.

4. *The Jacobian matrix evaluated at $x = 0$ is given by*

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{x=0} \\ &= \left.\begin{bmatrix} -1 & 9x_2^2 \\ -1 & -1 \end{bmatrix}\right|_{x=0} \\ &= \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \end{aligned}$$

and the eigenvalues are calculated as

$$\lambda_{1,2} = \{-2, 0\}$$

Using Lyapunov's indirect method results in no conclusion. Using phase plane analysis results in no conclusion.

Solution 2

The 2×2 system where

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

is given by

$$x^T M x = m_{11}x_1^2 + m_{21}x_1x_2 + m_{12}x_1x_2 + m_{22}x_2^2$$

Taking the time derivative of this system results in

$$\begin{aligned}
\frac{d}{dt}\left(x^T M x\right) &= 2m_{11}x_1\dot{x}_1 + 2m_{22}x_2\dot{x}_2 \\
&\quad + m_{21}\dot{x}_1x_2 + m_{21}x_1\dot{x}_2 + m_{12}\dot{x}_1x_2 + m_{12}x_1\dot{x}_2 + 2m_{22}x_2\dot{x}_2 \\
&= x_1\left(m_{11}+m_{11}\right)\dot{x}_1 + x_2\left(m_{22}+m_{22}\right)\dot{x}_2 \\
&\quad + x_2\left(m_{21}+m_{12}\right)\dot{x}_1 + x_1\left(m_{21}+m_{12}\right)\dot{x}_2 \\
&= x^T\left(M+M^T\right)\dot{x} \\
&= \dot{x}^T\left(M+M^T\right)x
\end{aligned}$$

When M is symmetric, it can be seen that

$$\begin{aligned}
\frac{d}{dt}\left(x^T M x\right) &= x^T\left(M+M^T\right)\dot{x} \\
&= x^T\left(M+M\right)\dot{x} \\
&= x^T 2M\dot{x} \\
&= 2x^T M\dot{x}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\left(x^T M x\right) &= \dot{x}^T\left(M+M^T\right)x \\
&= \dot{x}^T\left(M+M\right)x \\
&= \dot{x}^T 2Mx \\
&= 2\dot{x}^T Mx
\end{aligned}$$

Solution 3

1. *The system is given by*

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2^2 \\
\dot{x}_2 &= -x_2
\end{aligned}$$

where it can be seen that the equilibrium points are given by $(x_1^*, x_2^*) = (0,0)$. A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T P x, \quad P = P^T$$

which is positive definite if and only if all the leading principal minors of P are positive

$$\begin{aligned} p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0 \end{aligned}$$

(and it follows that $p_{22} > 0$). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x \\ &= \begin{bmatrix} -x_1 + x_2^2 \\ -x_2 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 + x_2^2 \\ -x_2 \end{bmatrix}^T \begin{bmatrix} p_{11}x_1 + p_{12}x_2 \\ p_{12}x_1 + p_{22}x_2 \end{bmatrix} \\ &= \left(-x_1 + x_2^2\right)(p_{11}x_1 + p_{12}x_2) - x_2(p_{12}x_1 + p_{22}x_2) \\ &= p_{12}x_2^3 - p_{11}x_1^2 - p_{22}x_2^2 - 2p_{12}x_1x_2 + p_{11}x_1x_2^2 \end{aligned}$$

By choosing $p_{12} = 0$, the term x_2^3 and x_1x_2 vanishes and the derivative is rewritten as

$$\begin{aligned} \dot{V}(x) &= -p_{11}x_1^2 - p_{22}x_2^2 + p_{11}x_1x_2^2 \\ &= -p_{11}x_1^2 - (p_{22} - p_{11}x_1)\, x_2^2 \\ &= -p_{11}x_1^2 - p_{11}\left(\frac{p_{22}}{p_{11}} - x_1\right) x_2^2 \\ &< 0, \quad \forall \frac{p_{22}}{p_{11}} > x_1 \end{aligned}$$

Taking $D = \left\{ x \in \mathbb{R}^n \middle|\, x_1 < \frac{p_{22}}{p_{11}} \right\}$, where $\frac{p_{22}}{p_{11}}$ may be chosen arbitrary large, shows that the equilibrium point is locally asymptotically stable.

2. *The system is given by*

$$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right) \\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right) \end{aligned}$$

where it can be seen that the equilibrium points are given by

$$(x_1^*, x_2^*) = (0, 0)$$

and the set

$$x_1^{*2} + x_2^{*2} = 1$$

This implies that the origin can not be globally asymptotically stable, since by starting the system in one of the points $x_1^{*2} + x_2^{*2} = 1$ *will keep the system in this point. A general quadratic Lyapunov function candidate is given by*

$$V(x) = \frac{1}{2}x^T P x, \quad P = P^T$$

which is positive definite if and only if all leading principal minors of P *have positive determinants, that is*

$$\begin{aligned} p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0 \end{aligned}$$

(and it follows that $p_{22} > 0$*). The derivative of the Lyapunov function candidate along the trajectories of the system is given by*

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^T P x \\
&= \begin{bmatrix} (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \left(2x_1x_2p_{12} - x_1x_2p_{11} + x_1x_2p_{22} + x_1^2p_{11} + x_1^2p_{12} - x_2^2p_{12} + x_2^2p_{22}\right)\left(x_1^2 + x_2^2 - 1\right) \\
&= x^T \begin{bmatrix} p_{11} + p_{12} & p_{12} - \frac{1}{2}p_{11} + \frac{1}{2}p_{22} \\ p_{12} - \frac{1}{2}p_{11} + \frac{1}{2}p_{22} & p_{22} - p_{12} \end{bmatrix} x \left(x_1^2 + x_2^2 - 1\right) \\
&= x^T Q x \left(x_1^2 + x_2^2 - 1\right)
\end{aligned}$$

By choosing Q *such that* $x^T Q x > 0 \quad \forall x \neq 0$ *and taking* $D = \{x \in \mathbb{R}^2 | \, x_1^2 + x_2^2 < 1\}$, *it can be seen that*

$$\dot{V}(x) < 0 \quad \forall x \in D$$

Choosing $p_{12} = 0$, *the matrix* P *is positive definite if and only if*

$$\begin{aligned} p_{11} &> 0 \\ p_{22} &> 0 \end{aligned}$$

and the matrix Q *is positive definite if and only if*

$$\begin{aligned} p_{11} &> 0 \\ p_{22} &> 0 \end{aligned}$$

by which it can be concluded that the origin of the system is asymptotically stable.

3. *The system is given by*

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2\\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

where it can be seen that the equilibrium point is given by

$$(x_1^*, x_2^*) = (0, 0)$$

A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T P x, \quad P = P^T$$

which is positive definite if and only if all the leading principal minors of P are positive, that is

$$\begin{aligned}p_{11} &> 0\\ p_{11}p_{22} - p_{12}^2 &> 0\end{aligned}$$

(and it follows that $p_{22} > 0$). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x\\ &= \begin{bmatrix} -x_1 - x_2 \\ x_1 - x_2^3 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\\ &= -p_{11}x_1x_2 - p_{12}x_1x_2 + p_{22}x_1x_2 - p_{11}x_1^2 + p_{12}x_1^2 - p_{12}x_2^2 - p_{22}x_2^4 - p_{12}x_1x_2^3\\ &= -(p_{11} - p_{12})x_1^2 - (p_{11} + p_{12} - p_{22})x_1x_2 - p_{12}x_2^2 - p_{22}x_2^4 - p_{12}x_1x_2^3\end{aligned}$$

In order to eliminate the undesirable terms, p_i is chosen according to

$$\begin{aligned}p_{11} + p_{12} - p_{22} &= 0\\ p_{12} &= 0\\ &\Rightarrow p_{11} = p_{22}\end{aligned}$$

which fulfills the requirements imposed in order to guarantee $V(x)$ positive definite. The derivative of $V(x)$ is now found as

$$\begin{aligned}\dot{V}(x) &= -p_{11}x_1^2 - p_{11}x_2^4\\ &< 0 \quad \forall x \in \mathbb{R}^2 - \{0\}\end{aligned}$$

Since $V(x)$ is radially unbounded, it can be concluded that the origin is globally asymptotically stable.

4. *The system is given by*

$$\begin{aligned}\dot{x}_1 &= -x_1 + 3x_2^3 \\ \dot{x}_2 &= -x_2 - x_1\end{aligned}$$

where it can be seen that the equilibrium point is given by

$$(x_1^*, x_2^*) = (0, 0)$$

Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}p_1 x_1^2 + \frac{1}{4}p_2 x_1^4 + \frac{1}{2}p_3 x_2^2 + \frac{1}{4}p_4 x_2^4$$

The derivative is found as

$$\begin{aligned}\dot{V}(x) &= p_1 x_1 \dot{x}_1 + p_2 x_1^3 \dot{x}_1 + p_3 x_2 \dot{x}_2 + p_4 x_2^3 \dot{x}_2 \\ &= \left(p_1 x_1 + p_2 x_1^3\right)\left(-x_1 + 3x_2^3\right) + \left(p_3 x_2 + p_4 x_2^3\right)\left(-x_2 - x_1\right) \\ &= -p_1 x_1^2 - p_2 x_1^4 - p_3 x_2^2 - p_4 x_2^4 \\ &\quad - (p_4 - 3p_1)\, x_1 x_2^3 + p_2 x_1^3 x_2^3 - p_3 x_1 x_2\end{aligned}$$

By choosing

$$\begin{aligned}p_1 &= \frac{1}{3}p_4 \\ p_2 &= 0 \\ p_3 &= 0 \\ p_4 &> 0\end{aligned}$$

it can be seen that

$$\begin{aligned}V(x) &= \frac{1}{6}p_4 x_1^2 + \frac{1}{4}p_4 x_2^4 \\ &> 0 \quad \forall x \in \mathbb{R}^2 - \{0\}\end{aligned}$$

and

$$\begin{aligned}\dot{V}(x) &= -\frac{1}{3}p_4 x_1^2 - p_4 x_2^4 \\ &< 0 \quad \forall x \in \mathbb{R}^2 - \{0\}\end{aligned}$$

Since The Lyapunov function is radially unbounded, it can be concluded that the origin of the system is globally asymptotically stable.

Solution 4

By using $\|x\|_4^4 = x_1^4 + x_2^4$ *it can be seen that*

$$\frac{1}{4}\|x\|_4^4 \leq V(x) \leq \frac{1}{4}\|x\|_4^4$$

The derivative $\dot{V}(x)$ *along the trajectories of the system is found as*

$$\begin{aligned}
\dot{V}(x) &= x_1^3\dot{x}_1 + x_2^3\dot{x}_2 \\
&= x_1^3\left(-x_2^3 - x_1\right) + x_2^3\left(x_1^3 - x_2\right) \\
&= -x_1^3x_2^3 - x_1^4 + x_2^3x_1^3 - x_2^4 \\
&= -x_1^4 - x_2^4 \\
&= -\|x\|_4^4
\end{aligned}$$

By Theorem 4.10, taking $k_1 = k_2 = \frac{1}{4}$ *and* $k_3 = 1$, *it can be concluded that the system is globally asymptotically stable.*

Solution 5

It can be seen that the function $V(x)$ *is not a Lyapunov function, however the function is radially unbounded. The derivative of* $V(x)$ *along the solutions of the system is given by*

$$\begin{aligned}
\dot{V}(x) &= x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - b)\dot{x}_2 \\
&= x_1(ax_1 - x_2x_1) + \frac{1}{\gamma}(x_2 - b)\gamma x_1^2 \\
&= ax_1^2 - x_2x_1^2 + x_2x_1^2 - bx_1^2 \\
&= -(b-a)x_1^2 \\
&\leq 0
\end{aligned}$$

Let $D = \mathbb{R}^2$ *and noticing that* $\Omega_c = \left\{x \in \mathbb{R}^2 |\, V(x) \leq c, \dot{V}(x) \leq 0\right\} = \{x \in \mathbb{R}^2 |\, V(x) \leq c\}$ *is a compact positively invariant and set for any finite* c *due to the radially unboundedness of* $V(x)$. *Let* $\Omega = \Omega_c$, *the set* E *is then found as* $E = \left\{x \in \Omega |\, \dot{V}(x) = 0\right\} = \{x \in \Omega |\, -(b-a)x_1^2 = 0\} = \{x \in \Omega |\, x_1 = 0\}$. *From the calculation of the equilibrium points it is known that* $x_1 = 0$ *is a invariant set. This implies that the largest invariant set in* E *is given by* $M = E$. *By Theorem 4.4 that every solution starting in* Ω *approaches* $x_1 = \{x \in \Omega |\, x_1 = 0\}$ *as* $t \to \infty$. *The steady state gain* k *is given by the value of* x_2 *when the system settles down, that is when* x_1 *reaches zero. The value of* k *will depend on the initial conditions, as illustrated in Figure 1.*

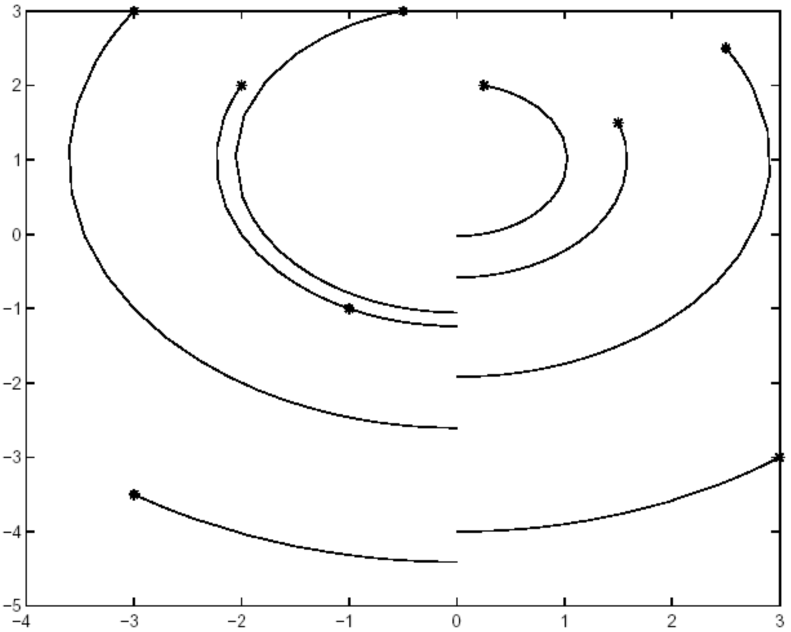

Figure 1: Simulation of the adaptive controller using $a = \gamma = 1$.

Solution 6
The system is given by

$$\begin{aligned}\dot{x}_1 &= 4x_1^2x_2 - f_1(x_1)\left(x_1^2 + 2x_2^2 - 4\right)\\ \dot{x}_2 &= -2x_1^3 - f_2(x_2)\left(x_1^2 + 2x_2^2 - 4\right)\end{aligned}$$

In order to show that $x_1^2 + 2x_2^2 - 4 = 0$ *is a invariant set, a new variable* $z = x_1^2 + 2x_2^2 - 4$ *is defined. The derivative of* z *is found as*

$$\begin{aligned}\dot{z} &= 2x_1\dot{x}_1 + 4x_2\dot{x}_2\\ &= 2x_1\left(4x_1^2x_2 - f_1(x_1)\left(x_1^2 + 2x_2^2 - 4\right)\right)\\ &\quad +4x_2\left(-2x_1^3 - f_2(x_2)\left(x_1^2 + 2x_2^2 - 4\right)\right)\\ &= -2x_1f_1(x_1)\left(x_1^2 + 2x_2^2 - 4\right) - 4x_2f_2(x_2)\left(x_1^2 + 2x_2^2 - 4\right)\\ &= -\left(2x_1f_1(x_1) + 4x_2f_2(x_2)\right)\left(x_1^2 + 2x_2^2 - 4\right)\\ &= -2\left(x_1f_1(x_1) + 2x_2f_2(x_2)\right)z\end{aligned}$$

where it can be seen that $z = 0$ *is a equilibrium point for the system, and consequently a invariant set for the system. This implies that* $x_1^2+2x_2^2-4 = 0$ *is a invariant set for the system. Consider the function*

$$V(x) = \left(x_1^2 + 2x_2^2 - 4\right)^2$$

which is radially unbounded. The derivative of V is found as

$$\begin{aligned}\dot{V}(x) &= 2\left(x_1^2+2x_2^2-4\right)\left(2x_1\dot{x}_1+4x_2\dot{x}_2\right)\\ &= -4\left(x_1f_1(x_1)+2x_2f_2(x_2)\right)\left(x_1^2+2x_2^2-4\right)^2\\ &\leq 0\end{aligned}$$

since $x_1f_1(x_1)$ and $x_2f_2(x_2)$ are grater than or equal to zero. Let $D=\mathbb{R}^2$ and noticing that $\Omega_c=\left\{x\in\mathbb{R}^2|\,V(x)\leq c,\dot{V}(x)\leq 0\right\}=\left\{x\in\mathbb{R}^2|\,V(x)\leq c\right\}$ is a compact positively invariant set for any finite c due to the radially unboundedness of $V(x)$. Let $\Omega=\Omega_c$, the set E is then found as

$$\begin{aligned}E &= \left\{x\in\Omega|\,\dot{V}(x)=0\right\}\\ &= \left\{x\in\Omega|\,x_1^2+2x_2^2-4=0 \text{ or } \left(x_1f_1(x_1)+2x_2f_2(x_2)\right)=0\right\}\\ &= \left\{x\in\Omega|\,x_1^2+2x_2^2-4=0 \text{ or } x_1=x_2=0\right\}\end{aligned}$$

From the state space model it can be seen that $x_1=x_2=0$ is a equilibrium point for the system ($f_1(0)=f_2(0)=0$). This implies that the largest invariant set in E is given by

$$M=\left\{x_1^2+2x_2^2-4=0\right\}\cup\left\{x_1=x_2=0\right\}$$

By Theorem 4.4 it can be concluded that every solution starting in Ω approaches $x_1^2+2x_2^2=4$ or the origin as $t\to\infty$. By choosing for instance $\Omega=\Omega_{15}=\left\{x\in\mathbb{R}^2|\,V(x)\leq 15,\dot{V}(x)\leq 0\right\}$, it can be seen that

$$E=\left\{x\in\Omega|\,x_1^2+2x_2^2-4=0\right\}$$

and

$$M=\left\{x_1^2+2x_2^2-4=0\right\}$$

which by Theorem 4.4 implies that every solution starting in Ω approaches $x_1^2+2x_2^2=4$. However, the set $\{x_1^2+2x_2^2-4=0\}$ is not a limit cycle since it contains equilibrium points (for instance $(x_1^*,x_2^*)=\left(0,\pm\sqrt{2}\right)$).

Solution 7

The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(x_1+x_2)-h(x_1+x_2)\end{aligned}$$

Let

$$g(x)=\begin{bmatrix}\alpha x_1+\beta x_2\\ \gamma x_1+\delta x_2\end{bmatrix}$$

where the symmetry requirement imposes the limitations

$$\beta = \gamma$$

The derivative of V along the trajectories of the system is now given by

$$\begin{aligned}
\dot{V}(x) &= g(x) f(x) \\
&= \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \beta x_1 + \delta x_2 \end{bmatrix}^T \begin{bmatrix} x_2 \\ -(x_1 + x_2) - h(x_1 + x_2) \end{bmatrix} \\
&= (\alpha x_1 + \beta x_2) x_2 + (\beta x_1 + \delta x_2)(-(x_1 + x_2) - h(x_1 + x_2))
\end{aligned}$$

taking $\beta = \delta$

$$\begin{aligned}
\dot{V}(x) &= (\alpha x_1 + \beta x_2) x_2 + \beta (x_1 + x_2)(-(x_1 + x_2) - h(x_1 + x_2)) \\
&= (\alpha x_1 + \beta x_2) x_2 - \beta (x_1 + x_2)^2 - \beta (x_1 + x_2) h(x_1 + x_2) \\
&= \alpha x_1 x_2 + \beta x_2^2 - \beta \left(x_1^2 + 2x_1 x_2 + x_2^2\right) - \beta (x_1 + x_2) h(x_1 + x_2) \\
&= \alpha x_1 x_2 - \beta x_1^2 - \beta 2 x_1 x_2 - \beta x_2^2 - \beta (x_1 + x_2) h(x_1 + x_2) \\
&= -\beta x_1^2 - (2\beta - \alpha) x_1 x_2 - \beta (x_1 + x_2) h(x_1 + x_2)
\end{aligned}$$

taking $\beta = \frac{1}{2}\alpha$

$$\begin{aligned}
\dot{V}(x) &= \beta x_1^2 - \beta (x_1 + x_2) h(x_1 + x_2) \\
&< 0 \quad \forall x \in \mathbb{R}^2
\end{aligned}$$

The function V is now found as

$$\begin{aligned}
V(x) &= \int_0^{x_1} \alpha y_1 dy_1 \\
&\quad + \int_0^{x_2} (\gamma x_1 + \delta y_2) dy_2 \\
&= \alpha \left[\frac{1}{2} y_1^2\right]_0^{x_1} + \gamma x_1 [y_2]_0^{x_2} + \delta \left[\frac{1}{2} y_2^2\right]_0^{x_2} \\
&= \frac{1}{2}\alpha x_1^2 + \gamma x_1 x_2 + \frac{1}{2}\delta x_2^2 \\
&= \beta x_1^2 + \beta x_1 x_2 + \frac{\beta}{2} x_2^2 \\
&= x^T P x
\end{aligned}$$

where

$$P = \begin{bmatrix} \beta & \frac{\beta}{2} \\ \frac{\beta}{2} & \frac{\beta}{2} \end{bmatrix}$$

and

$$\begin{aligned} \beta &> 0 \\ \frac{\beta^2}{2} - \frac{\beta^2}{4} &> 0 \end{aligned}$$

which implies that $P > 0$ (and $V(x)$ is positive definite on $\mathbb{R}^2$ and radially unbounded). By Theorem 4.2 it is concluded that the origin is globally asymptotically stable.

Solution 8

1. *From the figure it can be seen that*

$$\begin{aligned} \dot{x}_1 &= -g(e) + 2x_2 - x_1 \\ \dot{x}_2 &= g(e) - x_2 \\ e &= -x_1 \end{aligned}$$

   and the system is given by

$$\begin{aligned} \dot{x}_1 &= x_1^3 + 2x_2 - x_1 \\ \dot{x}_2 &= -x_1^3 - x_2 \end{aligned}$$

2. *Clearly the function $V(x)$ is positive definite and radially unbounded. The derivative of $V(x)$ along the trajectories of the system is given by*

$$\begin{aligned} \dot{V}(x) &= \frac{1}{2}\dot{x}^T P x + \frac{1}{2} x^T P \dot{x} \\ &= -x_1^2 - x_2^2 - 2x_1^3 x_2 \\ &= -x_1^2 - x_2^2 - 2x^T \begin{bmatrix} 0 & \frac{1}{2}x_1^2 \\ \frac{1}{2}x_1^2 & 0 \end{bmatrix} x \\ &= -x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - x^T \begin{bmatrix} 0 & x_1^2 \\ x_1^2 & 0 \end{bmatrix} x \\ &= -x^T \begin{bmatrix} 1 & x_1^2 \\ x_1^2 & 1 \end{bmatrix} x \\ &= -x^T Q(x) x \end{aligned}$$

   where positive definiteness of $Q(x)$ implies that the origin is asymptotically stable. In order for $Q(x)$ to be positive definite, is is required that all its leading principal minors are positive. This imposes the requirements

$$\begin{aligned} 1 &> 0 \\ 1 - x_1^4 &> 0 \end{aligned}$$

Taking $D = \{x \in \mathbb{R}^2 | \, |x_1| < 1\}$ and applying Theorem 4.1, shows that the origin is asymptotically stable.

3. *Since $V(x)$ is radially unbounded it is known that the set $\Omega_c = \{x \in \mathbb{R}^2 | \, V(x) \leq c\}$, where c is chosen such that $|x_1| < 1 \quad \forall x \in \Omega_c$, is positively invariant. The constant c is found as*

$$\begin{aligned} c &= \min_{|x_1|=1} V(x) \\ &= \min_{|x_1|=1} x^T P x \\ &= \min_{|x_1|=1} \left(\frac{1}{2} x_1^2 + x_1 x_2 + \frac{3}{2} x_2^2 \right) \\ &= \min \begin{cases} \frac{1}{2} + x_2 + \frac{3}{2} x_2^2, & \forall x_1 = 0 \\ \frac{1}{2} - x_2 + \frac{3}{2} x_2^2, & x_1 = 0 \end{cases} \end{aligned}$$

where it can be seen that

$$\begin{aligned} \frac{\partial}{\partial x_2} \left(\frac{1}{2} + x_2 + \frac{3}{2} x_2^2 \right) &= 1 + 3x_2 \\ \frac{\partial}{\partial x_2} \left(\frac{1}{2} - x_2 + \frac{3}{2} x_2^2 \right) &= -1 + 3x_2 \end{aligned}$$

which implies that

$$\begin{aligned} c &= \min_{|x_1|=1} V(x) \\ &= \min V(x), \quad x \in \left\{ \left(-1, -\frac{1}{3}\right), \left(1, \frac{1}{3}\right) \right\} \\ &= \min \left\{ V\left(-1, -\frac{1}{3}\right), V\left(1, \frac{1}{3}\right) \right\} \\ &= \frac{1}{3} \end{aligned}$$

Taking $\Omega = \Omega_{\frac{1}{3}}$, $E = \left\{ x \in \Omega | \, \dot{V}(x) = 0 \right\} = (0,0) = M$ which by Theorem 4.4 concludes that Ω may be taken as a estimate of the region of attraction. The parameter of the ellipsoid is calculated according to

$$\frac{q_1}{\left(\sqrt{\frac{2c}{\lambda_1}}\right)^2} + \frac{q_2}{\left(\sqrt{\frac{2c}{\lambda_2}}\right)^2} = 1$$

where

$$\Lambda = \begin{bmatrix} 0.29289 & 0 \\ 0 & 1.7071 \end{bmatrix}$$
$$M = \begin{bmatrix} -0.92388 & 0.38268 \\ 0.38268 & 0.92388 \end{bmatrix}$$

and consequently $a = 2.27$ *and* $b = 0.39$ *in the* q *system. The angle* θ *between the systems are found as*

$$\begin{aligned} \theta &= \arccos(-0.92388) \\ &= 2.7489[rad] \\ &= 157.50[\deg] \end{aligned}$$

Figure 2 shows a plot of the region of attracting.

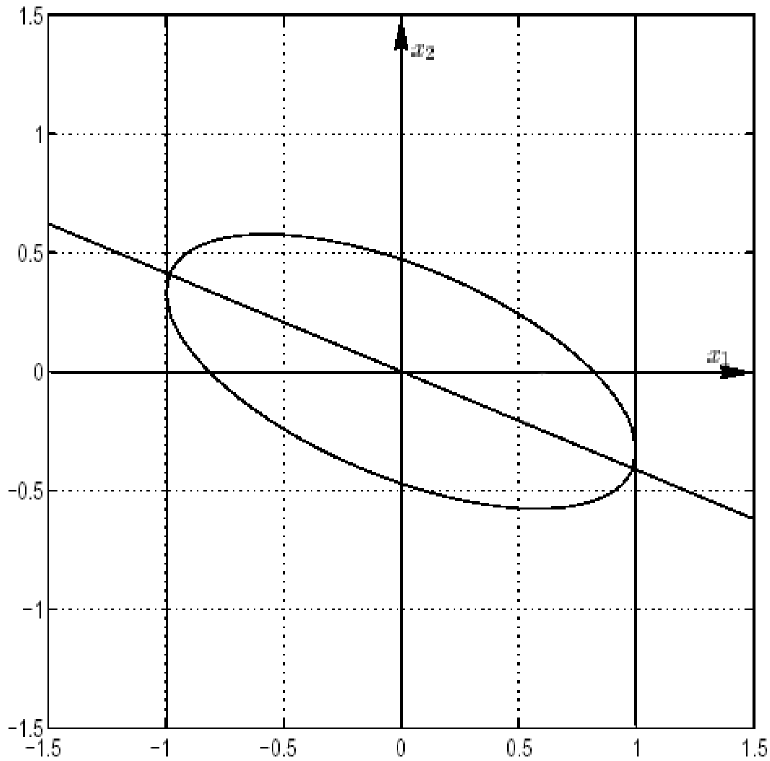

Figure 2: A estimate of the region of attraction