TTK4150 Nonlinear Control Systems
Solution 1

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Solution 1 (Exercise 3.2 in Khalil)
Notice that by definition D_r is a convex subset of $\mathbb{R}^n$.

1. *The pendulum equation with friction and constant input torque (Section 1.2.1) is given by*

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 + \frac{1}{ml^2}T \end{bmatrix} \tag{1}$$

The partial derivative of $f(x)$ with respect to x is found as

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -\frac{k}{m} \end{bmatrix} \tag{2}$$

From (1) and (2) it can be seen that $f(x)$ and $\frac{\partial f(x)}{\partial x}$ are continuous in x on $\mathbb{R}^2$. Using Lemma 3.1 or Lemma 3.2 it can be concluded that f is locally Lipschitz in x on D_r for any $r > 0$. Further it can be seen that $\frac{\partial f(x)}{\partial x}$ is bounded on $\mathbb{R}^2$, by which Lemma 3.3 concludes that f is globally Lipschitz (which also implies that f is locally Lipschitz in x on D_r for any $r > 0$).

2. *The tunnel diode circuit equation with constant input (Section 1.2.2) is given by*

$$f(x) = \begin{bmatrix} -\frac{1}{C}h(x_1) + \frac{1}{C}x_2 \\ -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u \end{bmatrix} \tag{3}$$

The partial derivative of $f(x)$ with respect to x is found as

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} -\frac{1}{C}\frac{\partial h(x_1)}{\partial x_1} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \tag{4}$$

From (3) and (4) it can be seen that $f(x)$ *and* $\frac{\partial f(x)}{\partial x}$ *are continuous in* x *on* $\mathbb{R}^2$ *(assuming* $h(x_1)$ *and* $\frac{\partial f(x)}{\partial x}$ *to be continuous in* x_1 *and* $h(x_1)$ *approaching* $\pm\infty$ *as* x_1 *approaches* $\pm\infty$*). Using Lemma 3.1 or Lemma 3.2 it can be concluded that* f *is locally Lipschitz in* x *on* D_r *for any finite* $r > 0$*. Further it can be seen that* $\frac{\partial f(x)}{\partial x}$ *is not globally bounded, by which Lemma 3.3 concludes that* f *is not globally Lipschitz.*

3. *The mass-spring equation with linear spring, linear viscous damping, Coulomb friction and zero external force (Section 1.2.3) is given by*

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}\eta(x_1, x_2) \end{bmatrix}$$

where $\eta(x_1, x_2)$ *is discontinuous at* $x_2 = 0$*. This discontinuity implies that* f *is not locally Lipschitz at* $x_2 = 0$ *(any discontinuous function is not locally Lipschitz at the point of discontinuity). Due to the definition of* D_r *(a ball centered around the origin) it is not possible to find a constant* r *small enough to make* f *locally Lipschitz in* x *on* D_r*, and it fallows that the system is not globally Lipschitz.*

4. *The Van der Pol oscillator (Example 2.6) is given by*

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 - \varepsilon\left(1 - x_1^2\right)x_2 \end{bmatrix} \tag{5}$$

The partial derivative of $f(x)$ *with respect to* x *is found as*

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & -\varepsilon\left(1 - x_1^2\right) \end{bmatrix} \tag{6}$$

From (5) and (6) it can be seen that $f(x)$ *and* $\frac{\partial f(x)}{\partial x}$ *are continuous in* x *on* $\mathbb{R}^2$*. Using Lemma 3.1 or Lemma 3.2 it can be concluded that* f *is locally Lipschitz in* x *on* D_r *for any finite* $r > 0$*. Further it can be seen that* $\frac{\partial f(x)}{\partial x}$ *is not globally bounded, by which Lemma 3.3 concludes that* f *is not globally Lipschitz.*

5. *The closed-loop equation of a third-order adaptive control system (Section 1.2.6), using* $x = \begin{bmatrix} e_0 & \phi_1 & \phi_2 \end{bmatrix}$*, is given by*

$$f(t, x) = \begin{bmatrix} a_m x_1 + k_p x_2 r(t) + k_p x_3\left(x_1 + y_m(t)\right) \\ -\gamma x_1 r(t) \\ -\gamma x_1\left(x_1 + y_m(t)\right) \end{bmatrix} \tag{7}$$

The partial derivative of $f(x)$ with respect to x is found as

$$\frac{\partial f(t,x)}{\partial x} = \begin{bmatrix} a_m + k_p x_3 & k_p r(t) & k_p \left(x_1 + y_m(t)\right) \\ -\gamma r(t) & 0 & 0 \\ -\gamma \left(2x_1 + y_m(t)\right) & 0 & 0 \end{bmatrix} \tag{8}$$

From (7) and (8) it can be seen that $f(t,x)$ and $\frac{\partial f(t,x)}{\partial x}$ are continuous in x on $[a,b] \times \mathbb{R}^3$ (assuming $r(t)$ and $y_m(t)$ bounded on $t \in [a,b]$). Using Lemma 3.1 or Lemma 3.2 it can be concluded that f is locally Lipschitz in x on $[a,b] \times D_r$ for any finite $r > 0$. Further it can be seen that $\frac{\partial f(t,x)}{\partial x}$ is not globally bounded, by which Lemma 3.3 concludes that f is not globally Lipschitz.

6. *By using the definition of Lipschitz on $f(x) = Ax - B\psi(Cx)$, the following is found*

$$\begin{aligned} \|f(x) - f(y)\| &= \|Ax - B\psi(Cx) - Ay + B\psi(Cy)\| \\ &= \|A(x-y) - B(\psi(Cx) - \psi(Cy))\| \\ &\leq \|A(x-y)\| + \|B(\psi(Cx) - \psi(Cy))\| \\ &\leq \|A\|\,\|x-y\| + \|B\|\,|\psi(Cx) - \psi(Cy)| \end{aligned} \tag{9}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz ($|f'(x)|$ is bounded by a positive constant k on $\mathbb{R}$). This implies that the inequality (9) may be rewritten as

$$\begin{aligned} \|f(x) - f(y)\| &\leq \|A\|\,\|x-y\| + \|B\|\,k\,\|x-y\| \\ &= (\|A\| + k\,\|B\|)\,\|x-y\| \\ &= L\,\|x-y\| \end{aligned}$$

Hence f is globally Lipschitz in x on $\mathbb{R}^n$, which implies that it is locally Lipschitz in x on D_r for any $r > 0$.

Solution 2 (Exercise 3.16 from Khalil)
The scalar differential equation is given by

$$\dot{x} = -x + \frac{\sin t}{1 + x^2}, x(0) = 2 \tag{10}$$

and

$$\frac{\partial f(t,x)}{\partial x} = -1 + \sin t \frac{2x}{\left(1 + x^2\right)^2}$$

which implies that f is locally Lipschitz in x on $[0,t] \times \mathbb{R}$ for any t since f and $\frac{\partial f(t,x)}{\partial x}$ are continuous on $[0,t] \times \mathbb{R}$ for any t (Lemma 3.2). By further *investigation it can be recognized that* f *is globally Lipschitz in* x *on* $[0,t]\times\mathbb{R}$ *for any* t ($\frac{\partial f(t,x)}{\partial x}$ *is uniformly bounded on* $[0,t]\times\mathbb{R}$, *Lemma 3.3). Using Theorem 3.2 it can be concluded that (10) has a unique solution* $x(t)$ *for all* $t\geq 0$.

In order to find a bound on the solution $x(t)$, *a new variable* v *is defined as*

$$v = x^2 \tag{11}$$

and it follows that $\dot{v}$ *may be upper bounded as*

$$\begin{aligned} \dot{v} &= 2x\dot{x} \\ &= -2x^2 + \sin t \frac{2x}{1+x^2} \\ &\leq -2x^2 + 1 \\ &= -2v + 1 \end{aligned} \tag{12}$$

where $v(0) = x(0)^2 = 4$. *It is easily seen that* $f(v) = -2v + 1$ *is globally Lipschitz in* v *on* $\mathbb{R}$ *by using the Lipschitz condition directly*

$$\begin{aligned} |f(x) - f(y)| &= |-2x + 1 + 2y - 1| \\ &= |-2x + 2y| \\ &= |-2(x-y)| \\ &= 2|x-y| \end{aligned} \tag{13}$$

Solving $\dot{u} = -2u + 1$, $u(0) = u_0$ *for* $t \geq 0$ *results in*

$$\begin{aligned} u(t) &= e^{-h(t)}\left(\int e^{h(t)}dt + c\right), \quad h(t) = \int 2dt = 2t \\ &= e^{-2t}\left(\int e^{2t}dt + c\right), \quad \int e^{2t}dt = \frac{1}{2}e^{2t} \\ &= e^{-2t}\left(\frac{1}{2}e^{2t} + c\right) \end{aligned}$$

where $u(0) = e^0\left(\frac{1}{2}e^0 + c\right) = \frac{1}{2} + c = u(0)$ *which with*

$$u(0) = v(0) = 4 \tag{14}$$

implies that

$$\begin{aligned} c &= u(0) - \frac{1}{2} = \frac{7}{2} \\ u(t) &= e^{-2t}\left(\frac{1}{2}e^{2t} + \frac{7}{2}\right) \end{aligned} \tag{15}$$

Using (12)-(15) and applying the comparison lemma (Lemma 3.4) results in

$$v(t) \leq e^{-2t}\left(e^{2t} + \frac{7}{2}\right) \quad t \geq 0 \tag{16}$$

since $u(t)$ *and* $v(t)$ *has a unique solution on* $\mathbb{R} \quad \forall t \geq 0$. *Finally it can be seen from combining (11) and (16) that* $x(t)$ *is bounded by*

$$\begin{aligned} |x(t)| &= \sqrt{v(t)} \\ &\leq \sqrt{\frac{7}{2}e^{-2t} + \frac{1}{2}} \quad t \geq 0 \end{aligned}$$

Solution 3

1. *The system is given by*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{16}x_1^5 - x_2 \end{bmatrix} \\ &= f(x) \end{aligned}$$

The equilibrium points are found by solving $0 = f(x^*)$

$$\begin{aligned} x_2^* &= 0 \\ -x_1^* + \frac{1}{16}x_1^{*5} - x_2^* &= 0 \end{aligned}$$

using $x_2^* = 0$, x_1^* *are found as*

$$\begin{aligned} -x_1^* + \frac{1}{16}x_1^{*5} &= 0 \\ -16x_1^* + x_1^{*5} &= 0 \\ x_1^*\left(x_1^{*4} - 16\right) &= 0 \\ &\Rightarrow x_1^* = \{-2, 0, 2\} \end{aligned}$$

which implies that the system has three isolated equilibrium points given by $(-2, 0)$, $(0, 0)$, *and* $(2, 0)$. *In order to determine the type of each isolated equilibrium point, the Jacobian of* $f(x)$ *is calculated*

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{5}{16}x_1^4 & -1 \end{bmatrix}$$

The eigenvalues and type of equilibrium are found as

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(-2,0)} &= \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \\ &\Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{17} \\ &\Rightarrow x^* = (-2, 0) \ \textit{ is a saddle} \end{aligned}$$

and

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(0,0)} &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \\ &\Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm j\frac{1}{2}\sqrt{3} \\ &\Rightarrow x^* = (-2,0) \text{ is a stable focus} \end{aligned}$$

and

$$\begin{aligned} \left.\frac{\partial f}{\partial x}\right|_{(2,0)} &= \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \\ &\Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{17} \\ &\Rightarrow x^* = (-2,0) \text{ is a saddle} \end{aligned}$$

2. *The system is given by*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2x_1 - x_1x_2 \\ 2x_1^2 - x_2 \end{bmatrix} \\ &= f(x) \end{aligned}$$

The equilibrium points are found by solving $0 = f(x^*)$

$$\begin{aligned} 2x_1^* - x_1^*x_2^* &= 0 \\ 2x_1^{*2} - x_2^* &= 0 \end{aligned}$$

using $x_2^* = 2x_1^{*2}$, x_1^* *is found as*

$$\begin{aligned} 2x_1^* - x_1^*x_2^* &= 2x_1^* - x_1^*2x_1^{*2} \\ &= 2x_1^*\left(1 - x_1^{*2}\right) \\ &= 0 \\ &\Rightarrow x_1^* = \{-1, 0, 1\} \end{aligned}$$

which implies that the system has three isolated equilibrium points given by $(-1,2)$, $(0,0)$, *and* $(1,2)$. *In order to determine the type of each isolated equilibrium point, the Jacobian of* $f(x)$ *is calculated*

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 2 - x_2 & -x_1 \\ 4x_1 & -1 \end{bmatrix}$$

The eigenvalues and type of equilibrium are found as

$$\begin{aligned}\left.\frac{\partial f}{\partial x}\right|_{(-1,2)} &= \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} \\ &\Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm j\frac{1}{2}\sqrt{15} \\ &\Rightarrow x^* = (-1,2) \ \textit{ is a stable focus}\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial f}{\partial x}\right|_{(0,0)} &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \\ &\Rightarrow \lambda_{1,2} = \{-1,2\} \\ &\Rightarrow x^* = (0,0) \ \textit{ is a saddle}\end{aligned}$$

and

$$\begin{aligned}\left.\frac{\partial f}{\partial x}\right|_{(1,2)} &= \begin{bmatrix} 0 & -1 \\ 4 & -1 \end{bmatrix} \\ &\Rightarrow \lambda_{1,2} = -\frac{1}{2} \pm j\frac{1}{2}\sqrt{15} \\ &\Rightarrow x^* = (-2,0) \ \textit{ is a stable focus}\end{aligned}$$

3. *The system is given by*

$$\begin{aligned}\dot{x} &= \begin{bmatrix} x_2 \\ -x_2 - \psi(x_1 - x_2) \end{bmatrix} \\ &= f(x)\end{aligned}$$

where

$$\psi(z) = \begin{cases} z^3 + \frac{1}{2}z, & |z| \le 1 \\ 2z - \frac{1}{2}sgn(z), & |z| > 1 \end{cases}$$

The equilibrium points are found by solving $0 = f(x^*)$

$$\begin{aligned}x_2^* &= 0 \\ -x_2^* - \psi(x_1^* - x_2^*) &= 0\end{aligned}$$

using $x_2^* = 0$, x_1^* *is found as*

$$\begin{aligned}\psi(x_1^*) &= 0 \\ &\Leftrightarrow x_1^{*3} + \frac{1}{2}x_1^*, \quad |x_1^*| \le 1 \\ &\Rightarrow x_1^* = 0\end{aligned}$$

which implies that the system has one isolated equilibrium point given by $(0,0)$. *In order to determine the type of equilibrium point, the eigenvalues of Jacobian of* $f(x)$ *is calculated at* x^*

$$\begin{aligned}
\left.\frac{\partial f}{\partial x}\right|_{x^*} &= \left.\begin{bmatrix} 0 & 1 \\ -\frac{1}{2} - 3\left(x_1 - x_2\right)^2 & -\frac{1}{2} + 3\left(x_1 - x_2\right)^2 \end{bmatrix}\right|_{x^*} \\
&= \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\
&\Rightarrow \lambda_{1,2} = -\frac{1}{4} \pm j\frac{1}{4}\sqrt{7} \\
&\Rightarrow x^* = (0,0) \text{ is a stable focus}
\end{aligned}$$

Solution 4

The phase portraits are drawn using the m-file pplane.m

1. *Figure 1 shows the display of pplane.m using* $x = x_1$ *and* $y = x_2$. *Figure*

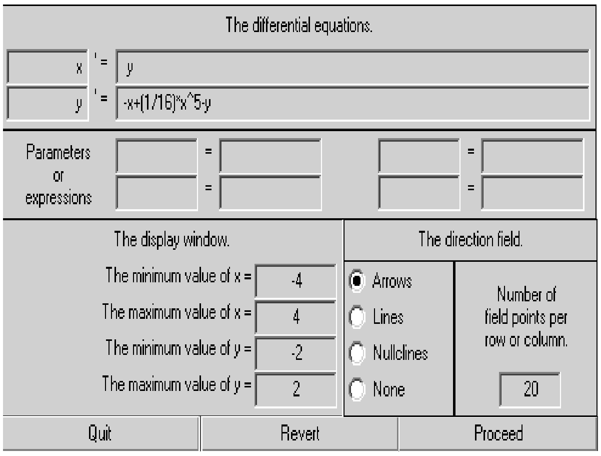

Figure 1: Display

2 shows the phase portrait the equilibrium points of the system. From the figure it can be seen that $(0,0)$ *has the property of a stable focus, and that* $(-2,0)$ *and* $(2,0)$ *have the property of a saddle point. This agrees with the findings in Exercise 3.*

2. *Figure 3 shows the display of pplane.m using* $x = x_1$ *and* $y = x_2$. *Figure 4 shows the phase portrait the equilibrium points of the system. From the figure it can be seen that* $(0,0)$ *has the property of a saddle point, and that* $(-1,2)$ *and* $(1,2)$ *have the property of stable node. This agrees with the findings in Exercise 3.*

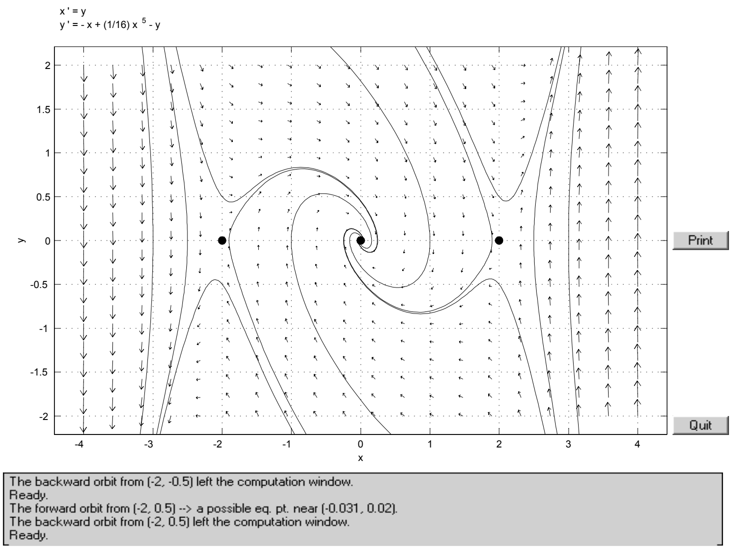

Figure 2: Phase portrat

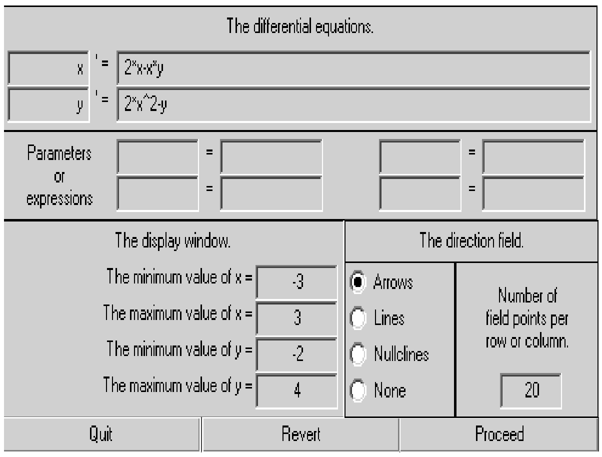

Figure 3: Display

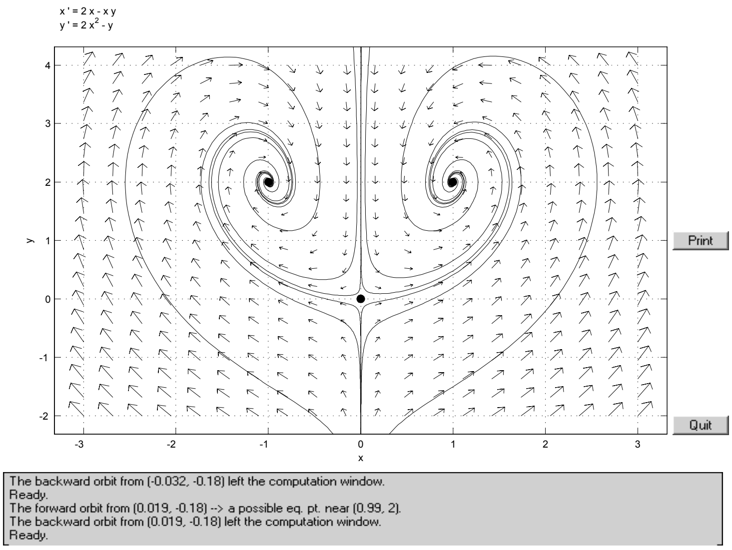

Figure 4: Phase portrait

Solution 5

The system is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -x_1 + x_2(x_1 + a) - b \\ -cx_1(x_1 + a) \end{bmatrix} \\ &= f(x) \end{aligned}$$

where $a, b, c > 0$, $b > a$ *and* $D = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_1 < -a \text{ and } x_2 < \frac{x_1+b}{x_1+a} \right\}$.

1. *Let*

$$V(x) = x_2 - \frac{x_1 + b}{x_1 + a}$$

where $V(x) = 0$ *on the boundary of the set* D. *Evaluating* $f(x) \bigtriangledown V(x)$ *yields*

$$\begin{aligned} f(x) \bigtriangledown V(x) &= f_1(x)\frac{\partial V(x)}{\partial x_1} + f_2(x)\frac{\partial V(x)}{\partial x_2} \\ &= f_1(x)\frac{a+b}{(x_1+a)^2} + f_2(x) \\ &= \frac{a+b}{(x_1+a)^2}\left(-x_1 + x_2(x_1+a) - b\right) - cx_1(x_1+a) \\ &= -cx_1(x_1+a) \quad \forall x_2 = \frac{x_1+b}{x_1+a} \\ &< 0 \quad \forall x \in \partial D \end{aligned}$$

where $\partial D = \left\{ x \in \mathbb{R}^2 \middle| x_2 = \frac{x_1+b}{x_1+a} \right\}$. *This implies that all the trajectories on the boundary of* D *move into* D*, by which it may be concluded that all trajectories starting in* D *stays in* D *for all future time.*

2. *Using Bendixson criterion*

$$\begin{aligned} \frac{f_1(x)}{\partial x_1} + \frac{f_2(x)}{\partial x_2} &= -1 + x_2 \\ &< -1 + \frac{x_1+b}{x_1+a} \quad \forall x \in D \\ &< 0 \quad \forall x \in D \end{aligned}$$

since $\lim_{x_1 \to -\infty} \frac{x_1+b}{x_1+a} \to 1$. *This implies that there can be no periodic orbits entirely in* D. *Using this and the fact that all trajectories starting in* D *remains in* D *for all future time, it can be concluded that there can be no periodic orbits passing through any point in* $x \in D$.

Solution 6

Suppose M *does not contain an equilibrium point. Then by the Poincare-Bendixson criterion, there is a periodic orbit in* M. *But, by Corollary 2.1, the periodic orbit must contain an equilibrium point.*

Solution 7

1. *By using* $x_1 = \delta$, $x_2 = \dot{\delta}$, $x_3 = E_q$, $u_1 = P$ *and* $u_2 = E_{FD}$ *the state space model is expressed as*

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} x_2 \\ -\frac{D}{M}x_2 - \frac{\eta_1}{M}x_3 \sin x_1 + \frac{1}{M}u_1 \\ -\frac{\eta_2}{\tau}x_3 + \frac{\eta_3}{\tau}\cos x_1 + \frac{1}{\tau}u_2 \end{bmatrix} \\ &= f(x) \end{aligned}$$

2. *Inserting the numerical values of the constants and the constant inputs in the state space model gives*

$$f(x) = \begin{bmatrix} x_2 \\ -4x_2 - 136.1x_3 \sin x_1 + 55.4 \\ -0.41x_3 + 0.26\cos x_1 + 0.18 \end{bmatrix}$$

The equilibrium points are found by solving $f(x^*) = 0$

$$\begin{aligned} x_2^* &= 0 \\ -4x_2^* - 136.1x_3^* \sin x_1^* + 55.4 &= 0 \\ -0.41x_3^* + 0.26\cos x_1^* + 0.18 &= 0 \end{aligned}$$

where it can be seen that

$$x_3^* = 0.63\cos x_1^* + 0.44$$

Using this and $x_2^* = 0$ *in* $4x_2^* - 136.1x_3^*\sin x_1^* + 55.4 = 0$ *results in*

$$-85.74\cos x_1^*\sin x_1^* - 59.88\sin x_1^* + 55.44 = 0$$

which has the solutions $x_1^* = \{0.41, 1.62\}$ *in the interval* $[-\pi, \pi]$ *(the other solutions correspond to one rotation relative to the solutions stated). Using* x_1^* *to calculate* x_3^* *the two equilibrium points are found as* $x^* = \{(0.41, 0, 1.01), (1.62, 0, 0.41)\}$. *Both the equilibrium points have* $\dot{\delta} = 0$, *implying that the system is at rest with respect to rotational velocity at the same frequency as the net frequency. The second equilibrium point has a relatively large angle and deviation from* $x_3 = 1$ *with respect to the first equilibrium point.*

Solution 8

1. *The state space is given by*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{I_z}\left(K\left(K_p\left(x_{10} - x_1\right) - T_d x_2\right) - f(x_2)\right) \end{bmatrix}$$

2. *By using* $f(x_2) = -x_2 + x_2|x_2|$ *and* $K = I_z = 1$, *the state space is rewritten as*

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ K_p\left(x_{10} - x_1\right) - T_d x_2 + x_2 - x_2|x_2| \end{bmatrix} \\ &= \begin{bmatrix} x_2 \\ K_p\left(x_{10} - x_1\right) - \left(T_d - 1\right)x_2 - x_2|x_2| \end{bmatrix} \end{aligned}$$

where the equilibrium point is given by

$$\begin{aligned} x_2^* &= 0 \\ K_p\left(x_{10} - x_1^*\right) - \left(T_d - 1\right)x_2^* - x_2^*|x_2^*| &= 0 \\ &\Rightarrow x_1^* = x_{10} \end{aligned}$$

3. *The Jacobian matrix evaluated at the equilibrium point is given by*

$$\begin{aligned} A &\triangleq \left.\frac{\partial f(x)}{\partial x}\right|_{x^*} \\ &= \left.\begin{bmatrix} 0 & 1 \\ -K_p & -\left(T_d - 1\right) - 2|x_2| \end{bmatrix}\right|_{x^*} \\ &= \begin{bmatrix} 0 & 1 \\ -K_p & -\left(T_d - 1\right) \end{bmatrix} \end{aligned}$$

where $\frac{\partial}{\partial x} x_2 |x_2|$ *is found by*

$$\begin{aligned} x_2 |x_2| &= \begin{cases} x_2^2, & \forall x_2 \geq 0 \\ -x_2^2, & \forall x_2 \geq 0 \end{cases} \\ &\Rightarrow \frac{\partial}{\partial x} x_2 |x_2| = \begin{cases} 2x_2, & \forall x_2 \geq 0 \\ -2x_2, & \forall x_2 \geq 0 \end{cases} = 2 |x_2| \end{aligned}$$

Notice that the type of equilibrium point is is independent of the set point x_{10}. *The eigenvalues are calculated*

$$\begin{aligned} eig(A) &= -\frac{1}{2} T_d \pm \frac{1}{2} \sqrt{-2T_d - 4K_p + T_d^2 + 1} + \frac{1}{2} \\ &= \frac{1}{2} \left(1 - T_d \pm \sqrt{-2T_d - 4K_p + T_d^2 + 1} \right) \\ &= \frac{1}{2} \left(1 - T_d \pm \sqrt{(T_d - 1)^2 - 4K_p} \right) \end{aligned}$$

From this expression the following conclusion can be drawn

(a) $K_p < \frac{1}{4} (T_d - 1)^2$ *results in two real eigenvalues*

(b) $K_p = \frac{1}{4} (T_d - 1)^2$ *results in two equal eigenvalues*

(c) $K_p > \frac{1}{4} (T_d - 1)^2$ *results in two complex conjugated eigenvalues*

Further it can be seen, by using $K_p, T_d > 0$, *that the only possible case for the eigenvalues to have a positive real part is when* $T_d < 1$ ($\sqrt{(T_d - 1)^2 - 4K_p} < \sqrt{(T_d - 1)^2} = |T_d - 1|$). *The result are shown graphically in Figure 5.*

4. *From Figure 5 it can be seen that* $K_p = T_d = 4$ *results in a stable focus. The system is given by*

$$\dot{x} = \begin{bmatrix} x_2 \\ K_p (x_{10} - x_1) - (T_d - 1) x_2 - x_2 |x_2| \end{bmatrix}$$

Using $K_p = T_d = 4$ *and* $x_{10} = 2$, *the system is rewritten as*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} x_2 \\ 4 (2 - x_1) - (4 - 1) x_2 - x_2 |x_2| \end{bmatrix} \\ &= \begin{bmatrix} x_2 \\ 4 (2 - x_1) - 3x_2 - x_2 |x_2| \end{bmatrix} \end{aligned}$$

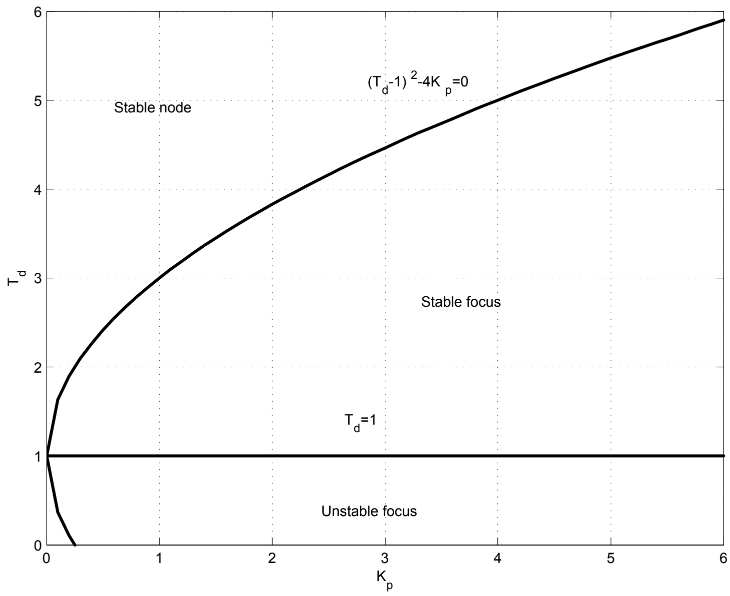

Figure 5: Type of equilibrium point

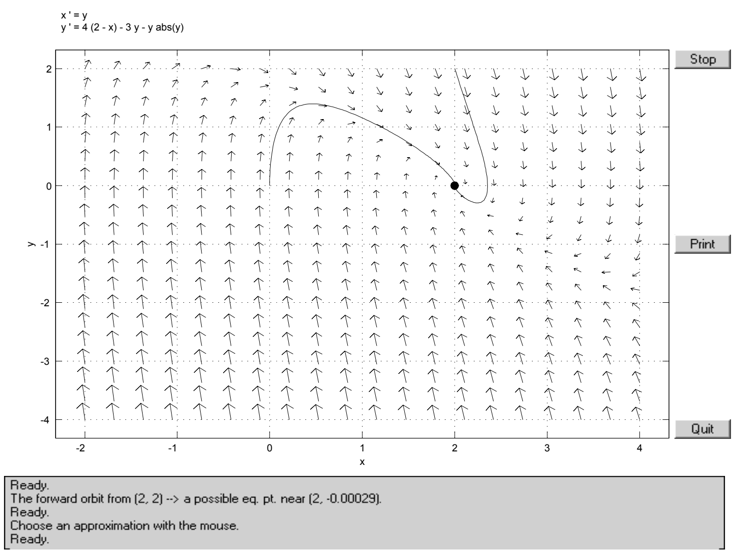

Figure 6: Trajectories in the phase plane when $K_p = T_d = 4$ and $x_{10} = 2$

Figure 6 shows the trajectories of the two different initial conditions (the figure was generated by pplane6.m). From this figure it seems like the equilibrium point is a stable node, this however is probably due trajectories starting too close to the equilibrium point in order to show focus behavior.