TTK4150 Nonlinear Control Systems
Exercise 2

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Exercise 1
Use Lyapunov's indirect method and the phase plane method to evaluate asymptotic stability and the type of equilibrium point of the origin in the following systems.

1.

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

2.

$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right) \\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right)\end{aligned}$$

3.

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

4.

$$\begin{aligned}\dot{x}_1 &= -x_1 + 3x_2^3 \\ \dot{x}_2 &= -x_2 - x_1\end{aligned}$$

Exercise 2
Given a constant matrix $M \in \mathbb{R}^{2\times 2}$ *and some time varying vector* $x \in \mathbb{R}^2$.

1. *Show that*

$$\frac{d}{dt}\left(x^T M x\right) = x^T\left(M + M^T\right)\dot{x}$$
$$= \dot{x}^T\left(M + M^T\right)x$$

2. *Assume further that M is symmetric ($M = M^T$), and show that*

$$\frac{d}{dt}\left(x^T M x\right) = 2x^T M \dot{x}$$
$$= 2\dot{x}^T M x$$

Remark 1
It can be shown that these results applies in the general case of $M \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$.

Exercise 3
Show that the origin of the following systems are asymptotically stable. Comment on local and global results.

1. *Use a quadratic Lyapunov function candidate on the system*

$$\dot{x}_1 = -x_1 + x_2^2$$
$$\dot{x}_2 = -x_2$$

2. *Use a quadratic Lyapunov function candidate on the system*

$$\dot{x}_1 = (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right)$$
$$\dot{x}_2 = (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right)$$

3. *Use a quadratic Lyapunov function candidate on the system*

$$\dot{x}_1 = -x_1 - x_2$$
$$\dot{x}_2 = x_1 - x_2^3$$

4. *Find a Lyapunov function for the system*

$$\dot{x}_1 = -x_1 + 3x_2^3$$
$$\dot{x}_2 = -x_2 - x_1$$

Exercise 4
Use the function $V(x) = \frac{1}{4}x_1^4 + \frac{1}{4}x_2^4$ to show that the origin of the system

$$\dot{x}_1 = -x_2^3 - x_1$$
$$\dot{x}_2 = x_1^3 - x_2$$

is globally exponentially stable.

Exercise 5

The system

$$\dot{y} = ay + u$$

is driven by an adaptive controller. The controller is given by

$$\begin{aligned} u &= -ky \\ \dot{k} &= \gamma y^2 \end{aligned}$$

where $\gamma > 0$ is a constant. By using $x_1 = y$ and $x_2 = k$ the closed loop state space model is given by

$$\begin{aligned} \dot{x}_1 &= ax_1 - x_2 x_1 \\ \dot{x}_2 &= \gamma x_1^2 \end{aligned}$$

From the state space it can be seen that $x_1 = 0$ is a invariant set (the equilibrium is a continuum of equilibrium points on the x_2-axis). Use the function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2$$

to show that the system trajectories converges to the invariant set when $t \to \infty$ when $b > a$. This implies that the adaptive controller manages to force y to zero. What can be said of the steady state gain, k, in the controller.

Exercise 6

Consider the system

$$\begin{aligned} \dot{x}_1 &= 4x_1^2 x_2 - f_1(x_1)\left(x_1^2 + 2x_2^2 - 4\right) \\ \dot{x}_2 &= -2x_1^3 - f_2(x_2)\left(x_1^2 + 2x_2^2 - 4\right) \end{aligned}$$

where the continuous functions f_1 and f_2 have the same sign as their arguments. Show that $x_1^2 + 2x_2^2 - 4 = 0$ and $x_1 = x_2 = 0$ are invariant sets, and that the solutions approaches these when $t \to \infty$. The set $x_1^2 + 2x_2^2 - 4 = 0$ is not a limit cycle. Why?

Exercise 7 (Exercise 4.6 in Khalil)

Exercise 8

Consider the system in Figure 1 where the nonlinear function is given by $g(e) = e^3$.

1. *Find the state space model.*

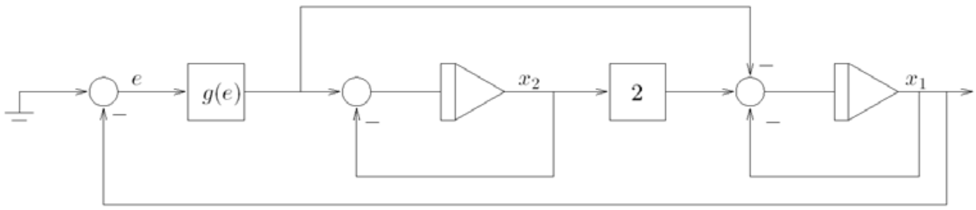

Figure 1: Block diagram of the system

2. *Show that the origin asymptotically stable using the Lyapunov function*

$$V(x) = x^T P x$$

where

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

3. *Sketch a estimate of the region of attraction in the* (x_1, x_2)*-plane.*