Seminar on tree products

An attempt at an overview

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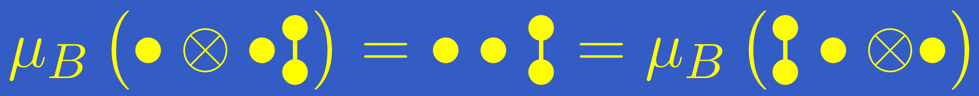
Introduction

We will here make an attempt at looking at different bialgebra structures on rooted trees. The covered types are

- Unordered trees (Butcher theory)
  - Juxtaposition product (trees $\mapsto$ forests) and coproduct using admissable cuts.
  - Duals of these and primitive elements.
- Ordered trees
  - Grossman-Larson product and coproduct
  - Duals.
- A connection between Butcher and Grossman-Larson?

Butcher-product

This is the bialgebra structure introduced in previous seminars on Hopf algebras.

- Product: $\mu_B(t_1 \otimes t_2) = t_1 t_2$ (unordered tuple of trees)
  - Multiplicative unit: $\varnothing$ (empty tree)
  - 
  - Commutative ($\mu_B \circ T = \mu_B$)
- Coproduct: $\Delta_B(t) = \sum_{\text{admissable cuts C of } t} P^C(t) \otimes R^C(t)$
  - 
  - Not cocommutative ($T \circ \Delta_B \neq \Delta_B$)

Butcher-product - Use

The current use of the Butcher-bialgebra structure is composition of B-series, that is, given two numerical methods ϕ_A and ϕ_B with corresponding coefficient functions $a : T \to \mathbf{R}$ and $b : T \to \mathbf{R}$, the B-series for the composed method $\phi_A \circ \phi_B$ is

$$B(b, B(a, y_0)) = B(a \star b, y_0)$$

where

$$a \star b(t) = \mu_{\mathbf{R}}(a \otimes b)\Delta_B(t)$$

Dualizing Butcher product (1/2)

For finding duals of the product and coproduct we use all trees and tuples of trees as an orthonormal basis.

The dual of the concatenation product is just all possible decomposition of the tuple t

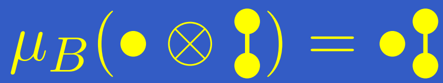
$$\mu_B^\star(t) = \sum_{\mu_B(t_1 \otimes t_2) = t} t_1 \otimes t_2$$

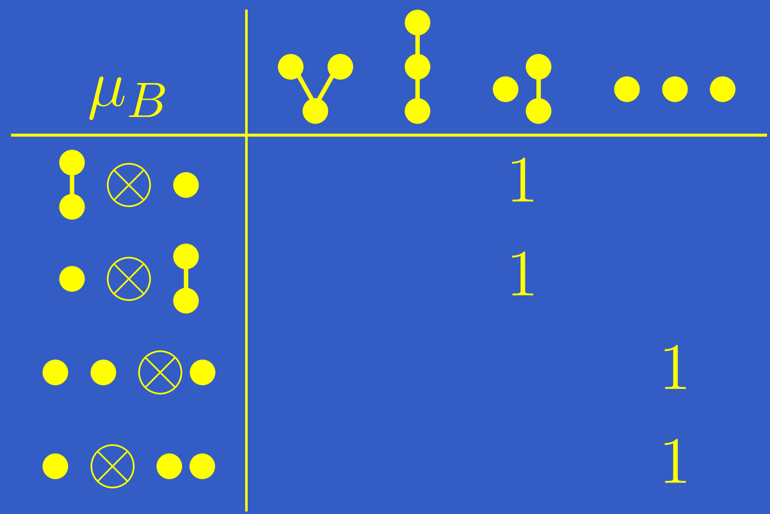
Examples

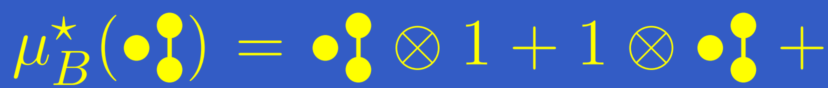
- $\mu_B^\star(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}) = \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} \otimes 1 + 1 \otimes \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}$
- $\mu_B^\star(\bullet\bullet) = \bullet\bullet \otimes 1 + 1 \otimes \bullet\bullet + \bullet \otimes \bullet$

All trees (not tuples) are primitive elements of $\mu_B^\star$.

Dualizing Butcher product (2/2)

Make a table of the product we are to dualize (with proper respect to the grading), e.g.  in row 2 here:

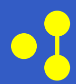


The dual of this product can now be found by reading down the columns of the table, e.g. 

Dualizing Butcher coproduct

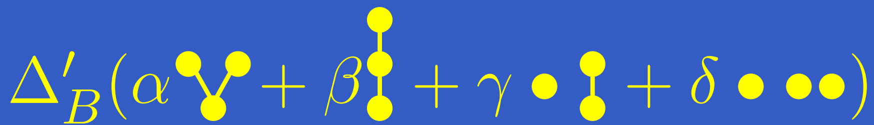
We can do the same thing to dualize the Butcher coproduct, here shown for grade 3.

Δ'_B	 $\otimes$ •	• $\otimes$ 	• • $\otimes$ •	• $\otimes$ ••
		2	1	
	1	1		
	1	1	1	1
• • •			3	3

so $\Delta^\star_B(\bullet\ \bullet \otimes \bullet) =$  $+$  $+\, 3$ • ••

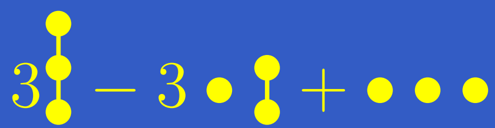
Primitive elements for Butcher coproduct (1/3)

Given the matrix on the previous slide, it is easy to find primitive elements of the Butcher coproduct, that is (tuples of) trees where $\Delta_B(t) = t \otimes 1 + 1 \otimes t$.

We need to solve  where the prime denotes that we omit terms including the identity tree 1.

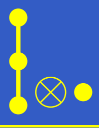
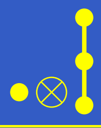
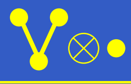
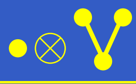
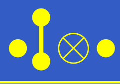
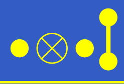
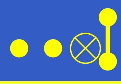
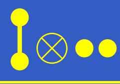
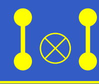
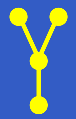
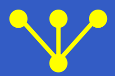
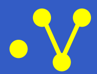
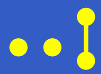
- *The nullspace of the transposed matrix gives these coefficients*

$\Longrightarrow$ The only primitive element of degree 3 is



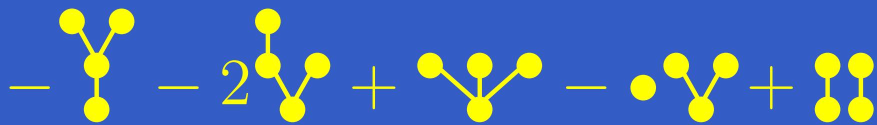
Primitive elements for Butcher coproduct (2/3)

Degree 4 primitive elements:

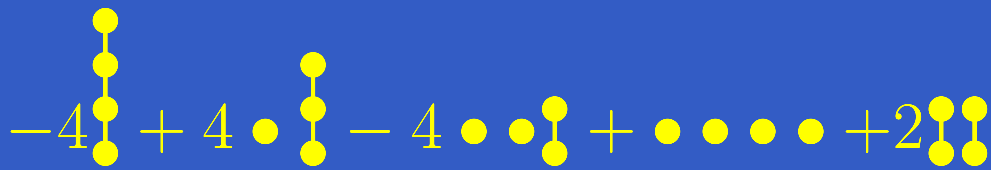
Δ'_B												
	1	1	0	0	0	0	0	0	0	0	1	0
	0	2	1	0	0	0	1	0	0	0	0	0
	0	1	0	1	1	0	1	0	0	0	1	0
	0	0	0	3	0	0	3	0	1	0	0	0
	0	0	1	1	0	2	2	0	1	0	0	1
	1	1	0	0	1	1	1	1	0	0	0	0
	0	0	0	0	2	2	1	1	1	1	0	2
	0	0	0	0	0	0	0	0	4	4	0	6
	0	0	0	0	2	2	0	0	0	0	2	1

Primitive elements for Butcher coproduct (3/3)

The primitive elements of degree 4 is then found be



and



For completeness, the only primitive elements of degree 2 and lower are

and

Grossman-Larson product

Cut off the root of the first tree, and spread out the subtrees onto the nodes of the second tree.

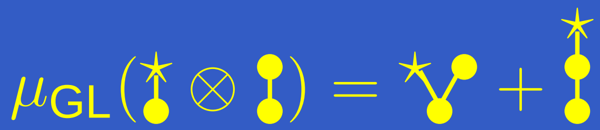
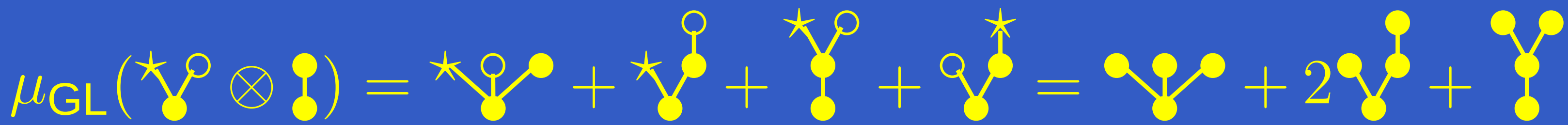
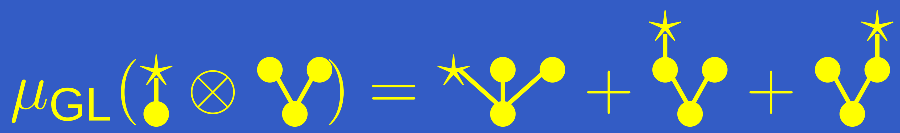
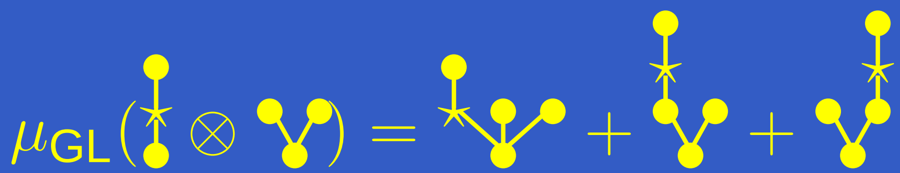
$$\mu_{\mathsf{GL}}(t_1 \otimes t_2) = \sum_{(d)} t_2 \leftharpoonup_d B^-(t_1)$$

sum over all mappings d: Subtrees$(t_1) \to$ Nodes(t_2)

- Multiplicative unit: • (both left and right)
- Ordering induced by ordering in t_1 and t_2. By convention, nodes from t_1 precede those of t_2 on each level.
- Not commutative
- Also called “the grafting product”

Grossman-Larson product examples

Examples

- 
- 
  (non-labelled trees)
- 
- 

Grossman-Larson coproduct

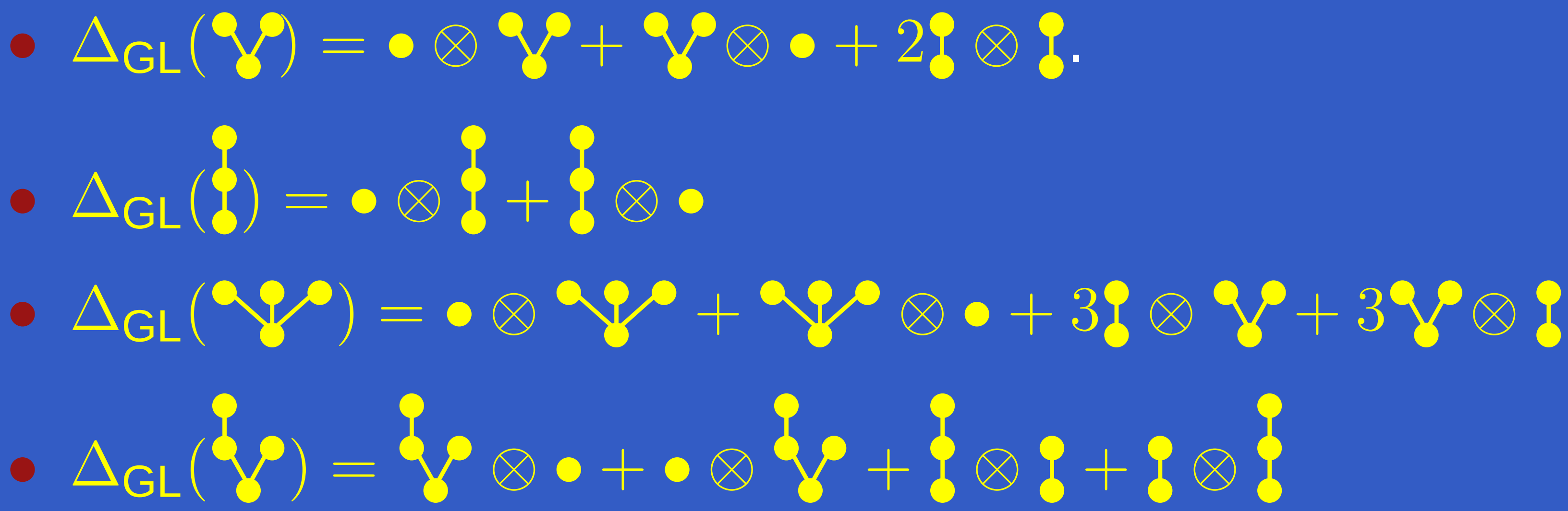
Let B^- be the operator mapping from a tree to the *ordered* tuple of the subtrees of t, and B^+ the operator mapping an ordered tuple to a tree by connecting all trees to a common root.

$$\Delta_{\mathsf{GL}}(t) = \sum_{\mathcal{X} \subseteq B^-(t)} B^+(\mathcal{X}) \otimes B^+(\mathcal{X}^C)$$

where $\mathcal{X}^C$ is the complement of $\mathcal{X}$ in $B^-(t)$. The sum includes the empty and the full set.

- This coproduct is cocommutative.

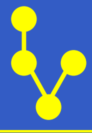
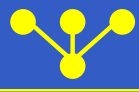
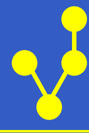
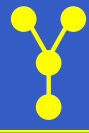
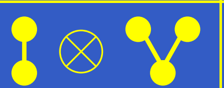
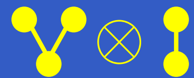
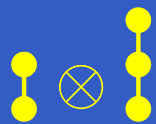
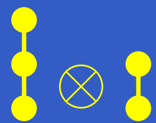
Grossman-Larson coproduct - examples

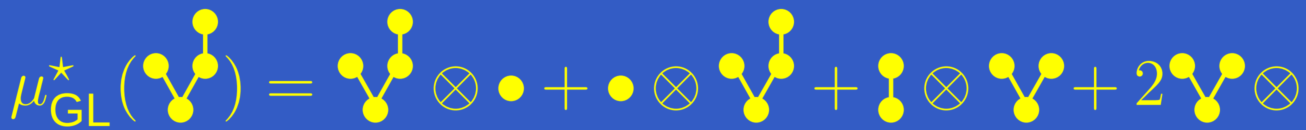


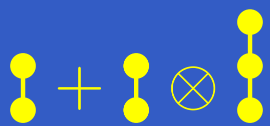
Note that trees with only one subtree from the root are primitive elements.

Dualizing Grossman-Larson product

For degree 4 (five nodes) we have the following table

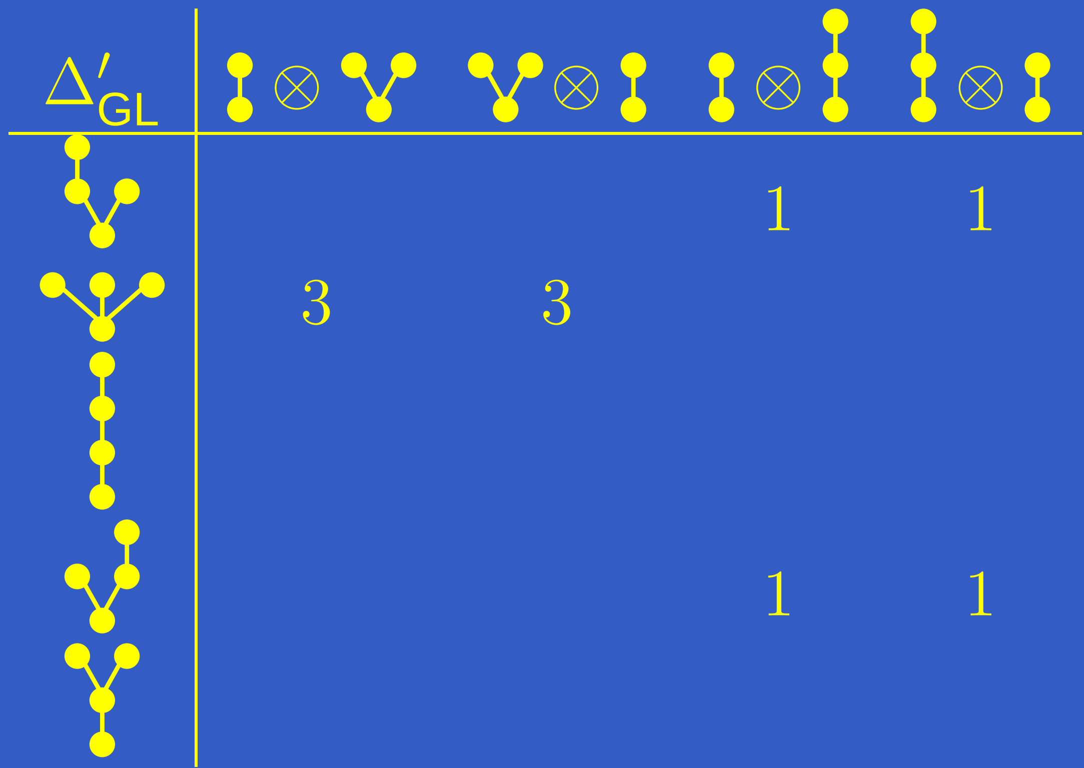
μ_{GL}					
	1	1		1	
		1		2	1
			1	1	1
	1		1		

with the result 



Dualizing Grossman-Larson coproduct

Using the same trees as in last slide (degree 4 - five nodes) we get



Grossman-Larson product in use (1/2)

μ_{GL} describes the product in the *algebra of elementary differential operators*. Let $\mathbb{F}(\bullet) = 1$ and define recursively

$$\mathbb{F}(B^+(t_1, \ldots, t_\nu)) = \sum_{i_1,\ldots,i_\nu} \mathbb{F}(t_1)[f_{i_1}] \cdots \mathbb{F}(t_\nu)[f_{i_\nu}] E_{i_1} \cdots E_{i_\nu}$$

We identify trees and operators as follows:

 $= \sum_{i,j} f_i f_j E_i E_j$ and  $= \sum_k f_k E_k$

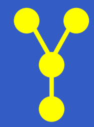
The tree $\mu_{\text{GL}}($  $\otimes$  $)$ should then characterize the operator

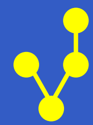
$$\sum f_i f_j E_i E_j \left[\sum f_k E_k \right]$$

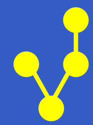
Grossman-Larson product in use (2/2)

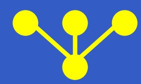
We skip all the $\sum$-symbols and calculate $f_i f_j E_i E_j \left[f_k E_k \right]$:

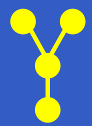
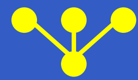
$$f_i f_j E_i E_j \left[f_k E_k \right] = f_i f_j E_i \left(E_j [f_k] E_k + f_k E_j E_k \right)$$

$$= f_i f_j E_i E_j [f_k] E_k$$ 

$$+ f_i f_j E_j [f_k] E_i E_k$$ 

$$+ f_i f_j E_i [f_k] E_j E_k$$ 

$$+ f_i f_j f_k E_i E_j E_k$$ 

which coincides with $\mu_{\mathrm{GL}}($  $\otimes$  $) =$  $+ 2$  $+$ .