Isotropy in geometric integration

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Introduction

We consider ambiguity in the formulation of Lie group methods. Examples used are

- The action of $SO(3)$ on the sphere S^2
- The action of $SL(2)$ on $\mathbf{R}^2$.

The ideas presented here are based on the paper by Lewis and Olver, *Geometric integration algorithms on homogeneous manifolds*, Found. Comput. Math. 2:363-392 (2002).

We shall see that the stability of first-order integrators can be significantly improved by use of the isotropy in the formulation.

Isotropy — definition

The isotropy subgroup of a Lie group action $\Lambda\colon G \times M \to M$ *is defined pointwise on the manifold as*

$$G_p = \{\, g \in G \mid \Lambda(g, p) = p \,\}$$

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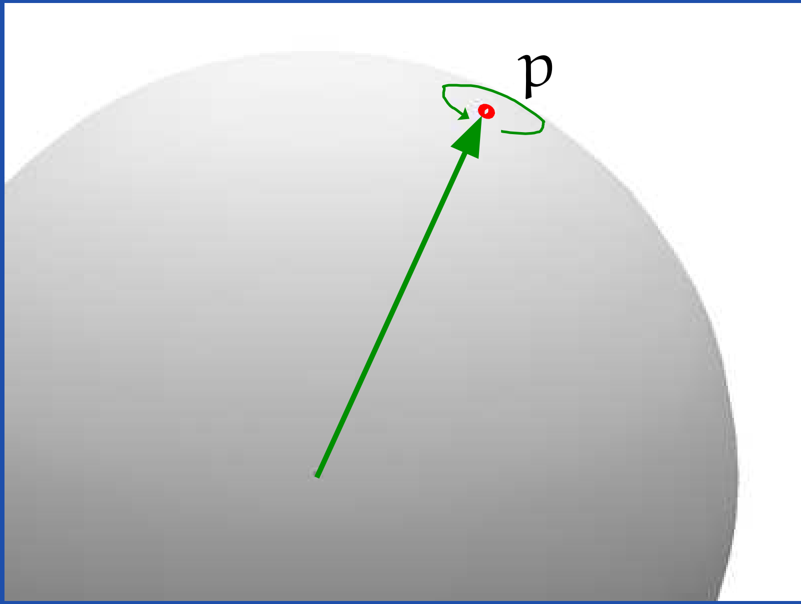
The isotropy subalgebra is the Lie algebra of G_p. Defining the Lie algebra action as $\lambda(u, p) = \Lambda(\exp(u), p)$, this is equivalent to

$$\mathfrak{g}_p = \{\, u \in \mathfrak{g} \mid \lambda(u, p) = p \,\}$$

Isotropy — $SO(3) - S^2$ example

Consider the action of $SO(3)$ on S^2, which rotates vectors in $S^2 \subset \mathbf{R}^3$.



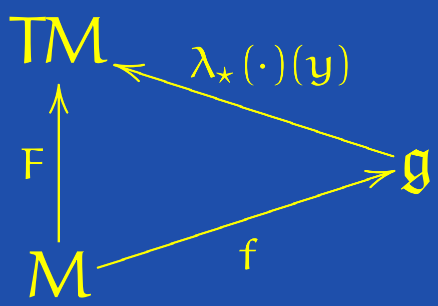
The isotropy subgroup of $SO(3)$ at the point p rotates p around itself.

Isotropy in RKMK methods

Given a differential equation on a manifold M

$$\dot{y} = F(y), \quad F : M \to TM$$

an RKMK method relies on the *existence* of an algebra-valued map $f \colon M \to \mathfrak{g}$ representing the differential equation. $\lambda_\star : \mathfrak{g} \times M \to TM$ is $\frac{d}{dt}\Big|_{t=0} \lambda(tu, y)$.



$$\lambda_\star(f(y))(y) = F(y)$$

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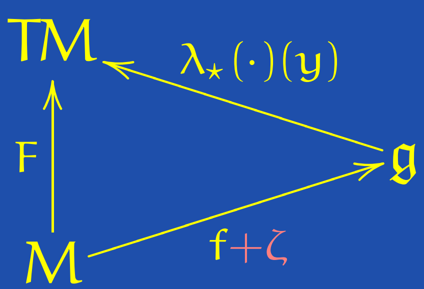

$$\lambda_\star(f(y) + \zeta(y))(y) = \underbrace{F(y)}_{\text{Not changed}}$$

where $\zeta(y)$ is any element in the isotropy subalgebra $\mathfrak{g}_y$.

Lie-Euler

The RKMK method we are going to use is Lie-Euler, which for some chosen algebra map f is

$$y_{n+1} = \exp(hf(y_n))y_n$$

where $f(y)y = F(y) = \dot{y}$.

(For our matrix-vector examples, $\lambda_\star(f(y))(y) = f(y)y$)

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Adding an isotropy correction to f preserves the differential equation, but affects the numerical method

$$y_{n+1} = \exp(h(f(y_n) + \sigma(y_n)\zeta(y_n)))y_n$$

where $\sigma : M \to \mathbf{R}$ is a scalar function.

Lie series expansion

An expansion of $\exp(f + \zeta)$ goes like (suppressing arguments of f and ζ)

$$\exp(f + \zeta) = I + f + \zeta + \frac{f^2 + f\zeta + \zeta f + \zeta^2}{2} + \cdots$$

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$$\exp(f+\zeta) = I + f + \cancel{\zeta} + \frac{f^2 + f\cancel{\zeta} + \zeta f + \cancel{\zeta^2}}{2} + \cdots$$

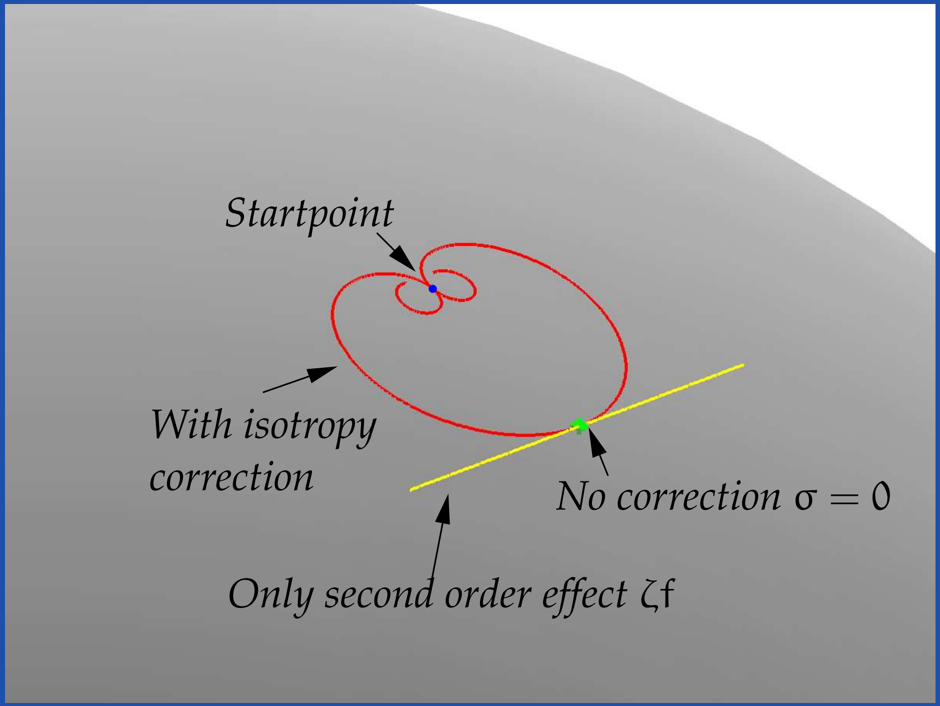
but remember that $\zeta y = 0$, so for Lie-Euler

$$\exp(f+\zeta)(y) = y + fy + \frac{f^2+\zeta f}{2}y + \frac{f^3 + f\zeta f + \zeta^2 f}{6}y + \cdots$$

$\Rightarrow$ isotropy only has an effect from second order ($\frac{h^2}{2}\zeta f$) and upwards.

The range of isotropy, $SO(3) - S^2$

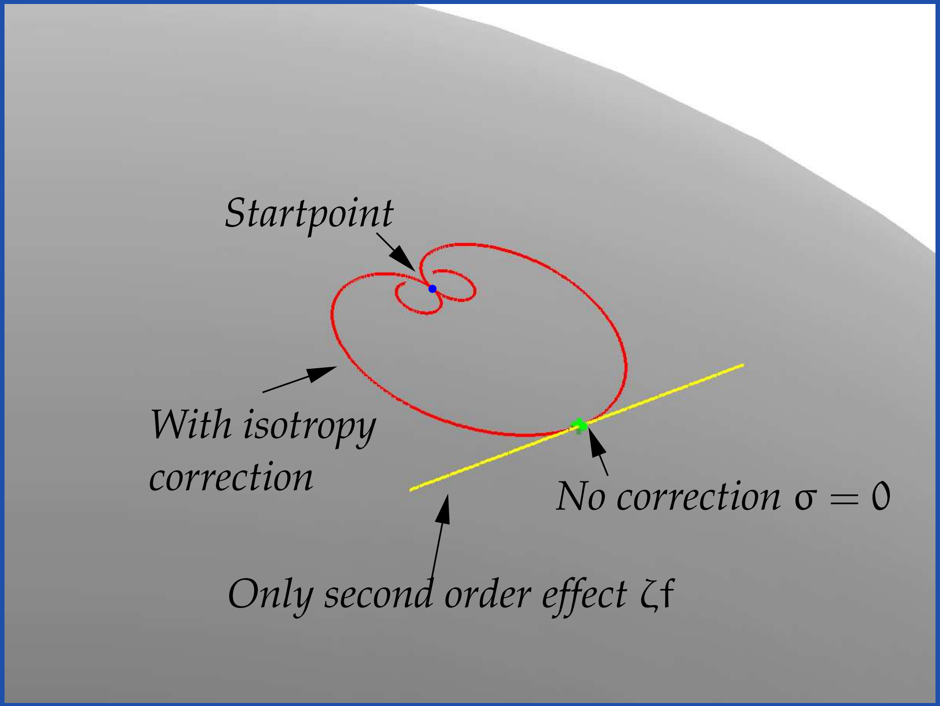
The effect of varying the scalar σ in front of ζ:



(One Lie-Euler step with isotropy correction, $\Delta t = 0.1$)

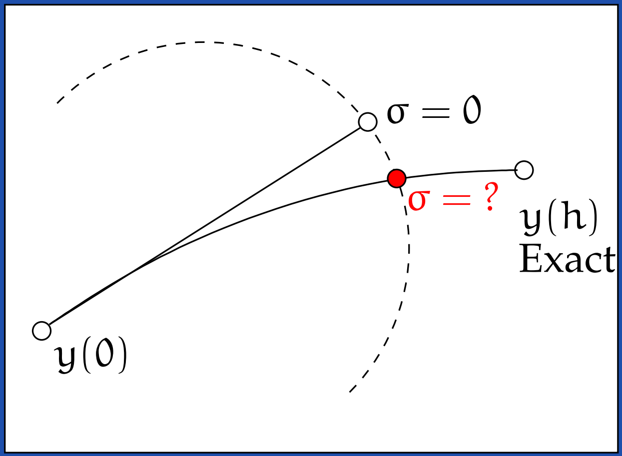
The range of isotropy, $SO(3) - S^2$

The effect of varying the scalar σ in front of ζ:



The second order effect ζf corrects the path orthogonally, as $\zeta f y \perp f y$ when ζ is skew-symmetric.

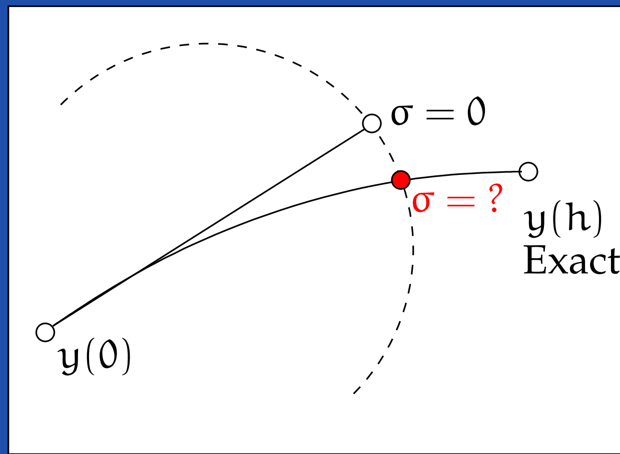
Orbit capture [Lewis-Olver]



Orbit capture is sought by choosing a σ such that we get close to the red point above.

- “Minimize the distance from the true orbit”

Orbit capture [Lewis-Olver]

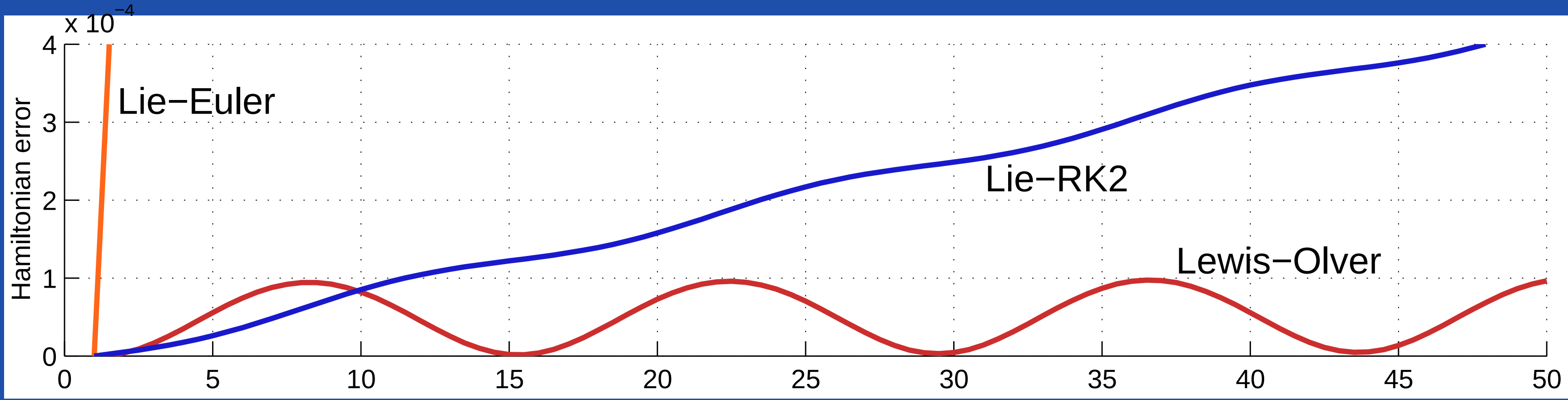


- “Minimize the distance from the true orbit”

By using isotropy, we are able to cancel the second order *orbit error*. Phase error is still order 1. Condition:

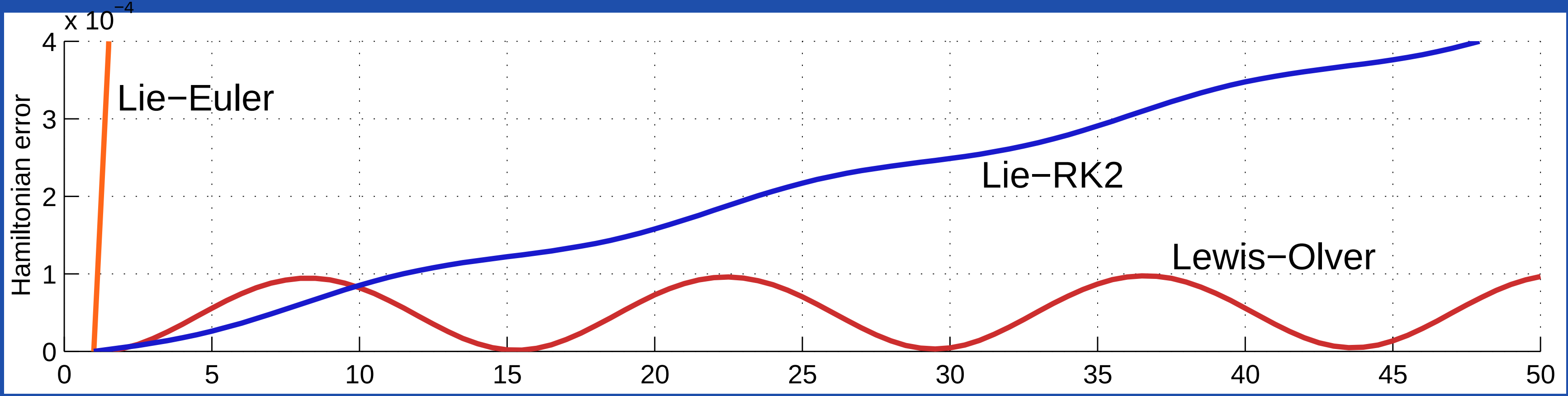
$$\underbrace{\frac{df(y)}{dt}y - \sigma\zeta f}_{\textit{second order error}} = C \underbrace{f(y)y}_{\textit{vector field}}$$

Results for the rigid body

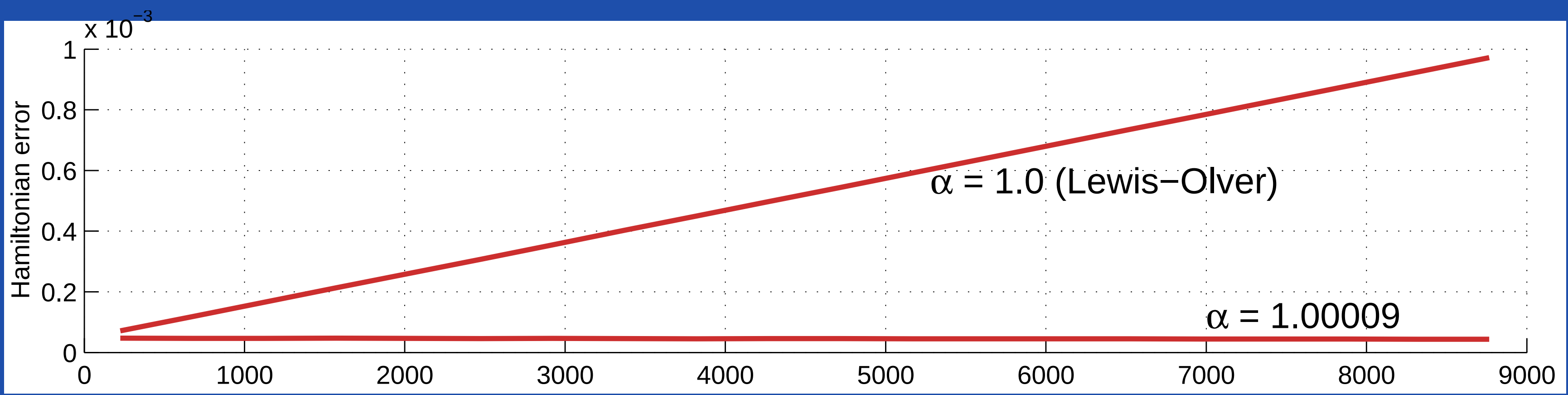


- The isotropy corrected Lie-Euler is signifi cantly better than no correction.
- There is some energy drift.

Results for the rigid body



- There is some energy drift.
- Remedy: Scale $\sigma(y)$ by a constant α



$SL(2)$ action on $\mathbf{R}^2$

$SL(2)$ is all 2×2 matrices with determinant 1.

We want a Lie-Euler method of the form

$$y_{n+1} = \exp(hf(y_n))y_n$$

where $f : \mathbf{R}^2 \to \mathfrak{sl}(2)$ *(trace-free matrices)*.

The isotropy subalgebra at $y = (u, v)$ in $\mathbf{R}^2$ is

$$\zeta(y) = \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}$$

$SL(2)$ action on $\mathbf{R}^2$

An $f\colon \mathbf{R}^2 \to \mathfrak{sl}(2)$ for *Lotka-Volterra*

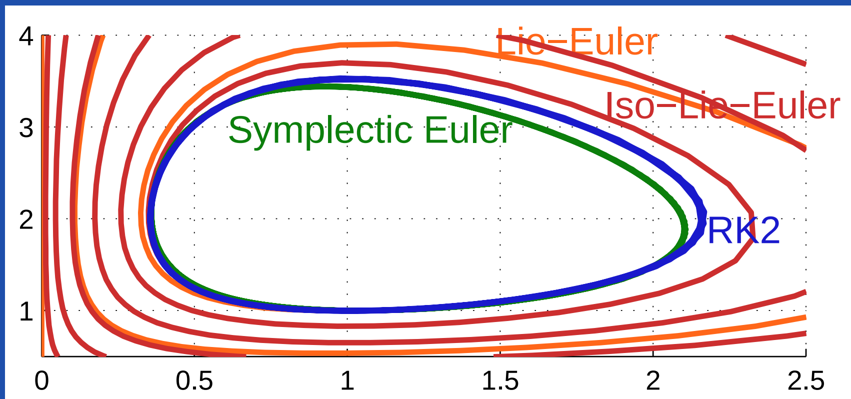
$$\left\{ \begin{aligned} \dot{u} &= u(v-2) \\ \dot{v} &= v(1-u) \end{aligned} \right\} \quad \Rightarrow \quad f(y) = \begin{pmatrix} u-1 & -\frac{u(u-v+1)}{v} \\ 0 & 1-u \end{pmatrix}$$

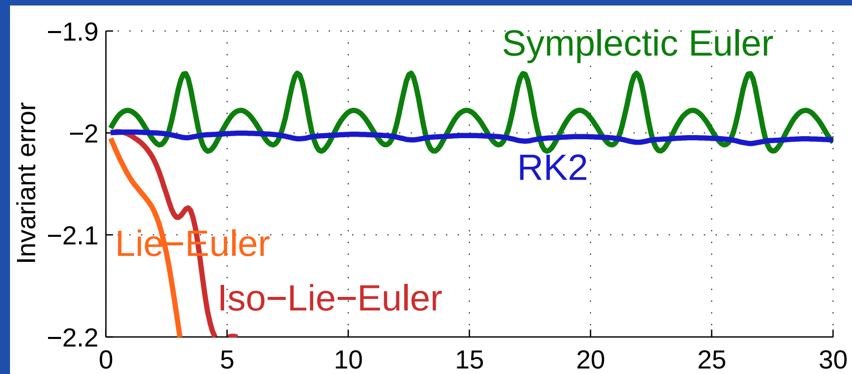
An $f\colon \mathbf{R}^2 \to \mathfrak{sl}(2)$ for *Duffing oscillator*

$$\left\{ \begin{aligned} \dot{u} &= v \\ \dot{v} &= u - u^3 \end{aligned} \right\} \quad \Rightarrow \quad f(y) = \begin{pmatrix} 0 & 1 \\ 1-u^2 & 0 \end{pmatrix}$$

Results on $SL(2)$, Lotka-Volterra, $\Delta t = 0.1$

No scaling, $\alpha = 1$





- Not as promising as the rigid body example

Results on $SL(2)$, Lotka-Volterra, $\Delta t = 0.1$

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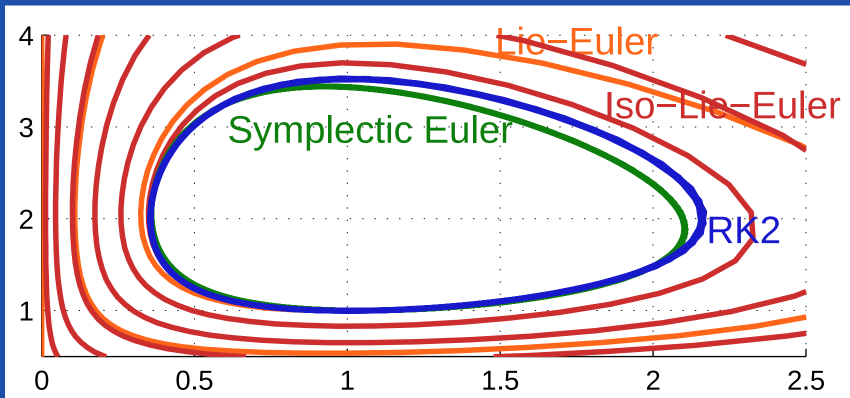
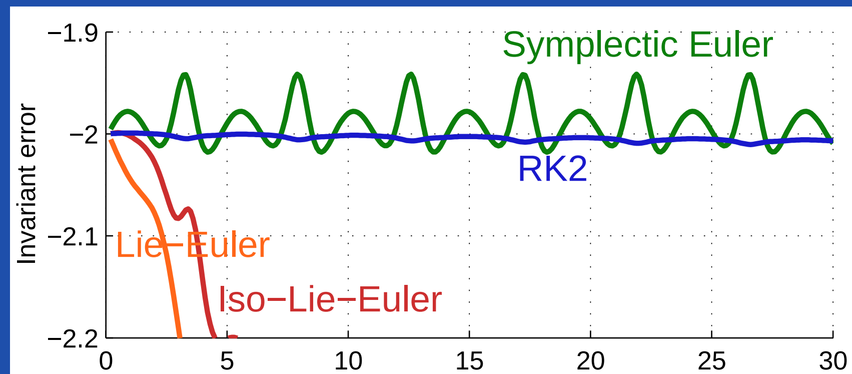
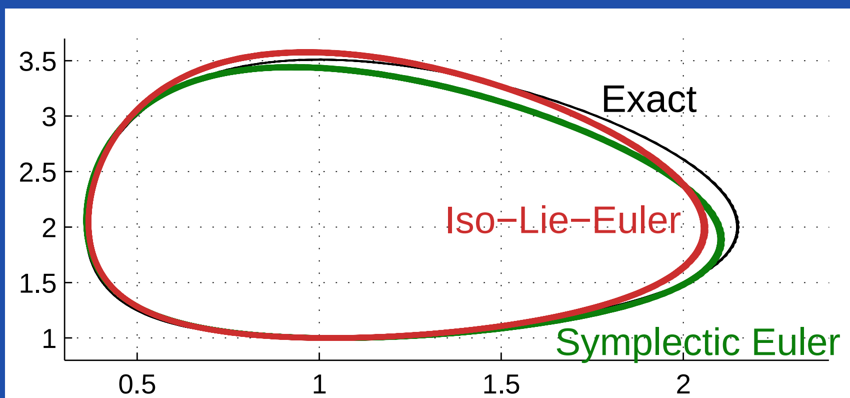
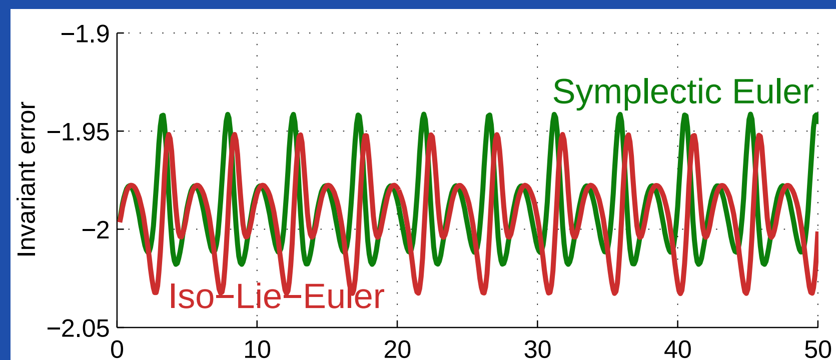




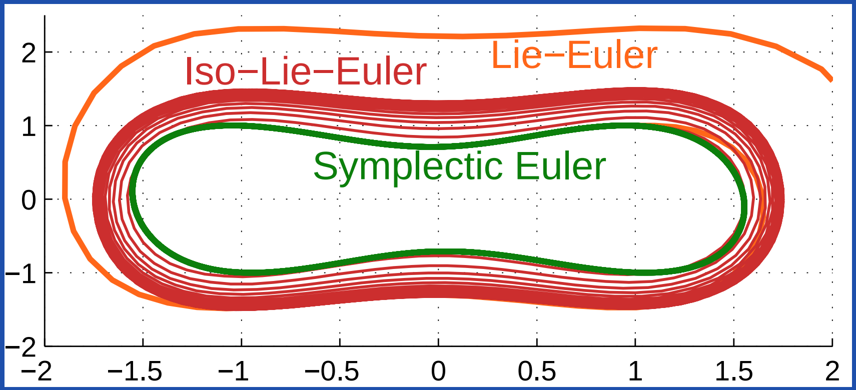
With scaling, $\alpha = 1.84$

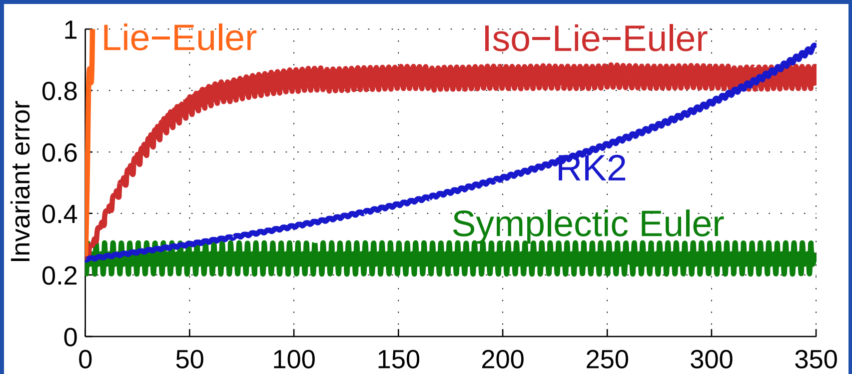




Results on $SL(2)$, Duffing, $\Delta t = 0.1$

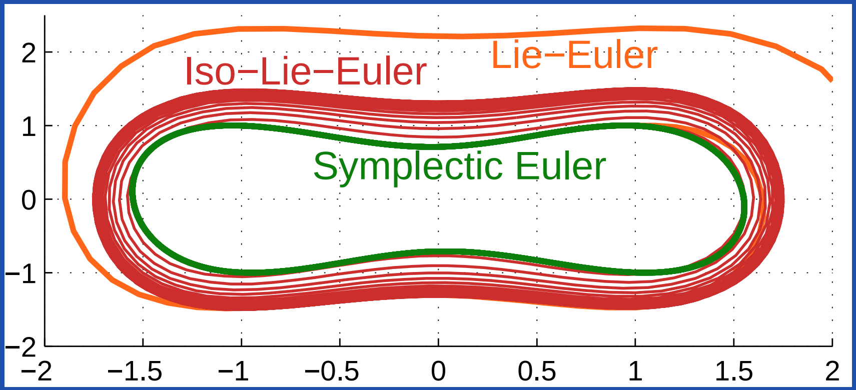
No scaling, $\alpha = 1$

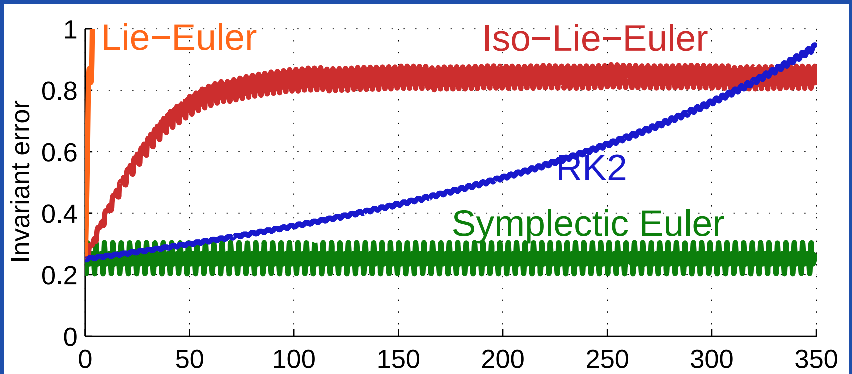




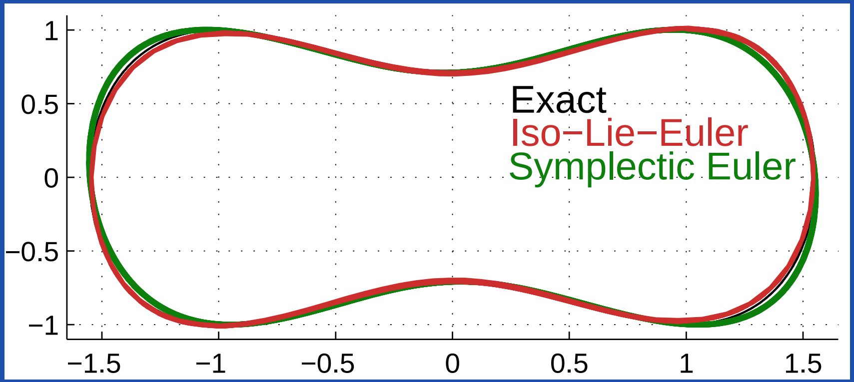
Results on $SL(2)$, Duffing, $\Delta t = 0.1$

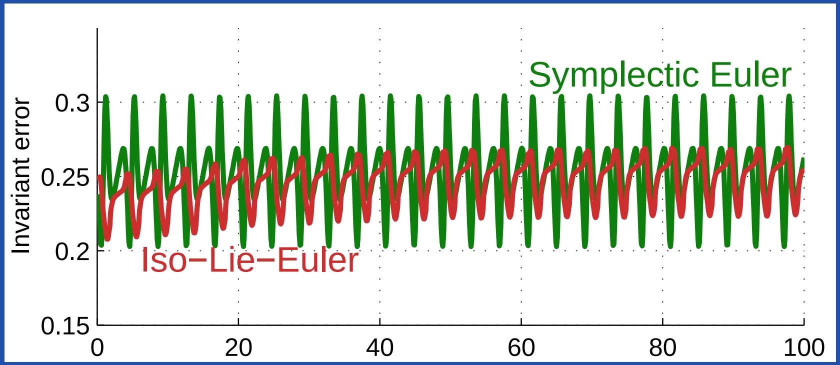
No scaling, $\alpha = 1$





With scaling, $\alpha = 1.17$



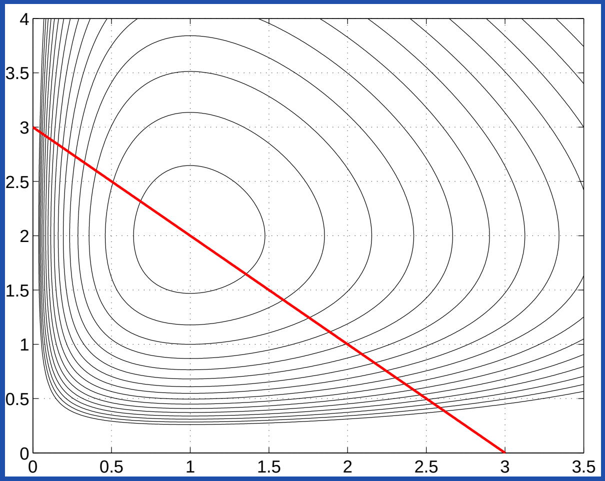


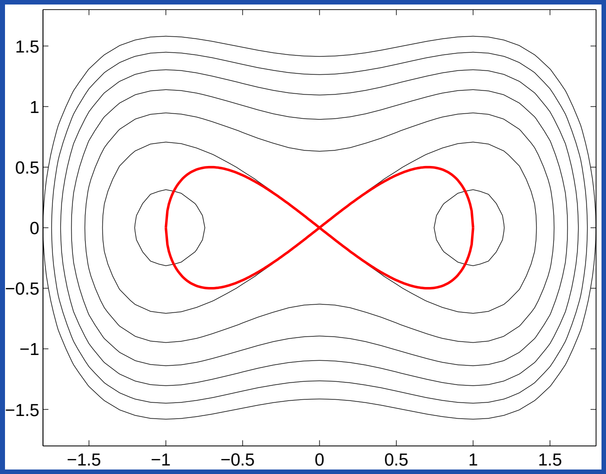
On the scaling α

- For rigid body: $\alpha = 1.00009$
- Lotka-Volterra: $\alpha = 1.84$, Duffing: $\alpha = 1.17$
- Found by trial and error.

On the scaling α

- For rigid body: $\alpha = 1.00009$
- Lotka-Volterra: $\alpha = 1.84$, Duffing: $\alpha = 1.17$
- Why small for rigid body?
- Partial answer: $\zeta f y \perp f y$ for rigid body, *not* true for our $SL(2)$ examples. We even have $\zeta f y \,\|\, f y$ at some points, which means that isotropy does not contribute here (in red below).





Notes

- Easily applicable to Lie-Euler on the rigid body equations, with good results.
- Stability comparable to symplectic euler when a satisfactory α has been found.
- Promising results recently noted for $SE(2)$.

The end

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Thank you for your attention