Isotropy in geometric integration

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Introduction

We consider ambiguity in the formulation of Lie group methods. Examples used are

- The action of $SO(3)$ on the sphere $S^2$
- The action of $SL(2)$ on $\mathbb{R}^2$.


We shall see that the stability of first-order integrators can be significantly improved by use of the isotropy in the formulation.
Isotropy — definition

The *isotropy subgroup* of a Lie group action \( \Lambda: G \times M \rightarrow M \) is defined pointwise on the manifold as

\[
G_p = \{ g \in G \mid \Lambda(g, p) = p \}
\]
Isotropy — definition

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\]

The *isotropy subalgebra* is the Lie algebra of \( G_p \). Defining the Lie algebra action as \( \lambda(u, p) = \Lambda(\exp(u), p) \), this is equivalent to

\[
g_p = \{ \, u \in g \mid \lambda(u, p) = p \, \}
\]
Isotropy — \( SO(3) \) — \( S^2 \) example

Consider the action of \( SO(3) \) on \( S^2 \), which rotates vectors in \( S^2 \subset \mathbb{R}^3 \).

The isotropy subgroup of \( SO(3) \) at the point \( p \) rotates \( p \) around itself.
Isotropy in RKMK methods

Given a differential equation on a manifold $\mathcal{M}$

$$\dot{y} = F(y), \quad F : \mathcal{M} \to T\mathcal{M}$$

an RKMK method relies on the existence of an algebra-valued map $f : \mathcal{M} \to g$ representing the differential equation. $\lambda_* : g \times \mathcal{M} \to T\mathcal{M}$ is $\frac{d}{dt}\bigg|_{t=0} \lambda(tu, y)$.

\[
\begin{array}{c}
\mathcal{M} \\
\downarrow f \\
g \\
\downarrow \lambda_*(\cdot)(y) \\
T\mathcal{M}
\end{array}
\]

\[
\lambda_*(f(y))(y) = F(y)
\]
Isotropy in RKMK methods

Given a differential equation on a manifold $M$

$$\dot{y} = F(y), \quad F: M \rightarrow \mathcal{T}M$$

an RKMK method relies on the existence of an algebra-valued map $f: M \rightarrow \mathcal{g}$ representing the differential equation. $\lambda_\star: \mathcal{g} \times M \rightarrow \mathcal{T}M$ is $\frac{d}{dt}\big|_{t=0} \lambda(tu, y)$, $\lambda_\star(\mathcal{g}_y)(y) = 0$.

$$\begin{array}{c}
\mathcal{T}M \\
\lambda_\star(\cdot)(y) \\
\downarrow F \\
\mathcal{g} \\
\downarrow \lambda_\star(f(y) + \zeta(y))(y) = F(y) \\
\text{Not changed}
\end{array}$$

where $\zeta(y)$ is any element in the isotropy subalgebra $\mathcal{g}_y$. 
The RKMK method we are going to use is Lie-Euler, which for some chosen algebra map $f$ is

$$y_{n+1} = \exp(hf(y_n))y_n$$

where $f(y)y = F(y) = \dot{y}$.

(For our matrix-vector examples, $\lambda_*(f(y))(y) = f(y)y$)
Lie-Euler

The RKMK method we are going to use is Lie-Euler, which for some chosen algebra map $f$ is

$$y_{n+1} = \exp(hf(y_n))y_n$$

where $f(y)y = F(y) = \dot{y}$.

Adding an isotropy correction to $f$ preserves the differential equation, but affects the numerical method

$$y_{n+1} = \exp(h(f(y_n) + \sigma(y_n)\zeta(y_n))))y_n$$

where $\sigma: \mathcal{M} \rightarrow \mathbb{R}$ is a scalar function.
Lie series expansion

An expansion of $\exp(f + \zeta)$ goes like (suppressing arguments of $f$ and $\zeta$)

$$\exp(f + \zeta) = I + f + \zeta + \frac{f^2 + f\zeta + \zeta f + \zeta^2}{2} + \ldots$$
Lie series expansion

An expansion of $\exp(f + \zeta)$ goes like (suppressing arguments of $f$ and $\zeta$)

$$\exp(f + \zeta) = I + f + \zeta + \frac{f^2 + f\zeta + \zeta f + \zeta^2}{2} + \cdots$$

but remember that $\zeta y = 0$, so for Lie-Euler

$$\exp(f + \zeta)(y) = y + fy + \frac{f^2 + \zeta f}{2} y + \frac{f^3 + f\zeta f + \zeta^2 f}{6} y + \cdots$$

$\Rightarrow$ isotropy only has an effect from second order ($\frac{1}{2} \zeta f$) and upwards.
The range of isotropy, $SO(3) - S^2$

The effect of varying the scalar $\sigma$ in front of $\zeta$:

(One Lie-Euler step with isotropy correction, $\Delta t = 0.1$)
The range of isotropy, $\mathbf{SO}(3) - S^2$

The effect of varying the scalar $\sigma$ in front of $\zeta$:

The second order effect $\zeta f$ corrects the path orthogonally, as $\zeta fy \perp fy$ when $\zeta$ is skew-symmetric.
Orbit capture is sought by choosing a $\sigma$ such that we get close to the red point above.

- “Minimize the distance from the true orbit”
Orbit capture [Lewis-Olver]

“Minimize the distance from the true orbit”

By using isotropy, we are able to cancel the second order orbit error. Phase error is still order 1. Condition:

\[
\frac{df(y)}{dt} y - \sigma \zeta f = C \quad f(y) y
\]

- second order error
- vector field
Results for the rigid body

- The isotropy corrected Lie-Euler is significantly better than no correction.
- There is some energy drift.
Results for the rigid body

- There is some energy drift.
- Remedy: Scale $\sigma(y)$ by a constant $\alpha$
SL(2) action on $\mathbb{R}^2$

$\text{SL}(2)$ is all $2 \times 2$ matrices with determinant 1.

We want a Lie-Euler method of the form

$$y_{n+1} = \exp(hf(y_n))y_n$$

where $f : \mathbb{R}^2 \to \text{sl}(2)$ (trace-free matrices).

The isotropy subalgebra at $y = (u, v)$ in $\mathbb{R}^2$ is

$$\zeta(y) = \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}$$
**SL(2) action on $\mathbb{R}^2$**

An $f: \mathbb{R}^2 \rightarrow \mathfrak{sl}(2)$ for *Lotka-Volterra*

\[
\begin{cases}
\dot{u} = u(v - 2) \\
\dot{v} = v(1 - u)
\end{cases} \quad \Rightarrow \quad f(y) = \begin{pmatrix} u - 1 & -\frac{u(u-v+1)}{v} \\ 0 & 1 - u \end{pmatrix}
\]

An $f: \mathbb{R}^2 \rightarrow \mathfrak{sl}(2)$ for *Duffing oscillator*

\[
\begin{cases}
\dot{u} = v \\
\dot{v} = u - u^3
\end{cases} \quad \Rightarrow \quad f(y) = \begin{pmatrix} 0 & 1 \\ 1 - u^2 & 0 \end{pmatrix}
\]
Results on $\mathbb{SL}(2)$, Lotka-Volterra, $\Delta t = 0.1$

No scaling, $\alpha = 1$

- Not as promising as the rigid body example
Results on $SL(2)$, Lotka-Volterra, $\Delta t = 0.1$

No scaling, $\alpha = 1$

With scaling, $\alpha = 1.84$
Results on $\text{SL}(2)$, Duffing, $\Delta t = 0.1$

No scaling, $\alpha = 1$

No scaling, $\alpha = 1$
Results on $\text{SL}(2)$, Duffing, $\Delta t = 0.1$

No scaling, $\alpha = 1$

With scaling, $\alpha = 1.17$
On the scaling $\alpha$

- For rigid body: $\alpha = 1.00009$
- Lotka-Volterra: $\alpha = 1.84$, Duffing: $\alpha = 1.17$
- Found by trial and error.
On the scaling \( \alpha \)

- For rigid body: \( \alpha = 1.00009 \)
- Lotka-Volterra: \( \alpha = 1.84 \), Duffing: \( \alpha = 1.17 \)
- Why small for rigid body?
- Partial answer: \( \zeta_{fy} \perp fy \) for rigid body, \textit{not} true for our \( SL(2) \) examples. We even have \( \zeta_{fy} \parallel fy \) at some points, which means that isotropy does not contribute here (in red below).
Easily applicable to Lie-Euler on the rigid body equations, with good results.

Stability comparable to symplectic euler when a satisfactory $\alpha$ has been found.

Promising results recently noted for $SE(2)$. 
The end

- Easily applicable to Lie-Euler on the rigid body equations, with good results.
- Stability comparable to symplectic euler when a satisfactory $\alpha$ has been found.
- Promising results recently noted for $SE(2)$.

Thank you for your attention