Isotropy in geometric integration

FoCM'02

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Introduction

The ideas are based on

- Lewis and Olver, *Geometric integration algorithms on homogeneous manifolds*, preprint, University of Minnesota, 2001.

Outline:

- A strategy to minimize local error by use of isotropy.
- Numerical results from the above strategy to ridid body rotations and two example differential equations on $\mathbf{R}^2$.
- A tweak by a constant gives extraordinary stability.

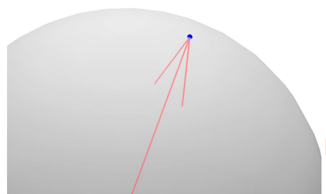
Isotropy

Definition: *The* isotropy subgroup *of a Lie group action* $\Lambda\colon G \times M \to M$ *is defined pointwise on the manifold as*

$$G_p = \{\, g \in G \mid \Lambda(g,p) = p \,\}$$

The isotropy subalgebra is the Lie algebra of G_p. Defining the Lie algebra action as $\lambda(u,p) = \Lambda(\exp(u),p)$, this is equivalent to

$$\mathfrak{g}_p = \{\, u \in \mathfrak{g} \mid \lambda(u,p) = p \,\}$$



Isotropy in RKMK methods

For the differential equation

$$\dot{y} = F(y), \quad F : M \to TM$$

an RKMK method relies on the existence of a $f : M \to \mathfrak{g}$

$$\lambda_*(f(y))(y) = F(y) \quad \text{for} \quad \lambda_* : \mathfrak{g} \times M \to TM$$

Let $\zeta(y)$ be any element in the isotropy subalgebra at y,

$$\lambda_*(f(y) + \zeta(y))(y) = F(y)$$

as well. f is thus *not unique!*

Lie-Euler

The RKMK method we are going to use is Lie-Euler, which uncorrected is

$$y_{n+1} = \exp(hf(y_n))y_n$$

and the corrected version is

$$y_{n+1} = \exp(h(f(y_n) + \sigma(y_n)\zeta(y_n)))y_n$$

where σ is a scalar function we are going to adjust in order to improve the consecutive steps of Lie-Euler.

Local isotropy correction

A Lie series expansion of the exact flow and the flow of corrected Lie-Euler leads to a second order condition (assuming matrix-vector group action):

$$\boxed{\sigma(y)\zeta(y)f(y)y = \frac{\mathrm{d}f(y)}{\mathrm{d}t}y} \qquad (*)$$

At each point we use numerical differentiation for the right side, and search for a σ which approximates $(*)$ “in some sense”.

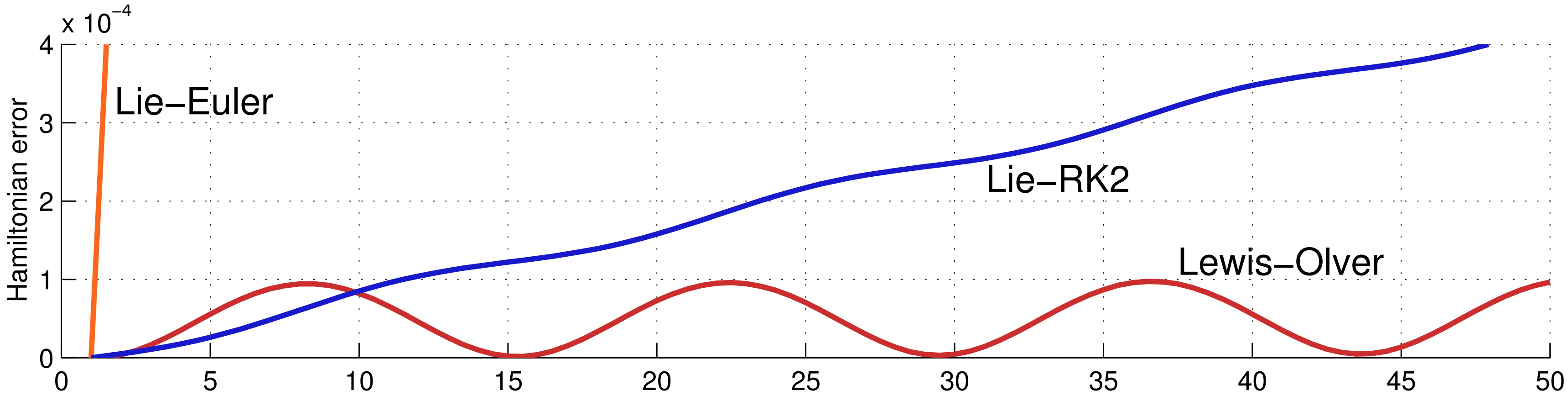
Isotropy for rigid body

The Lewis and Olver correction for $\dot{y} = F(y) = y \times \mathbb{I}^{-1} y$ is

$$\sigma(y) = \frac{\langle F(y), \mathbb{I}^{-1} F(y) \rangle}{\|F(y)\|^2} - \langle y, \mathbb{I}^{-1} y \rangle$$

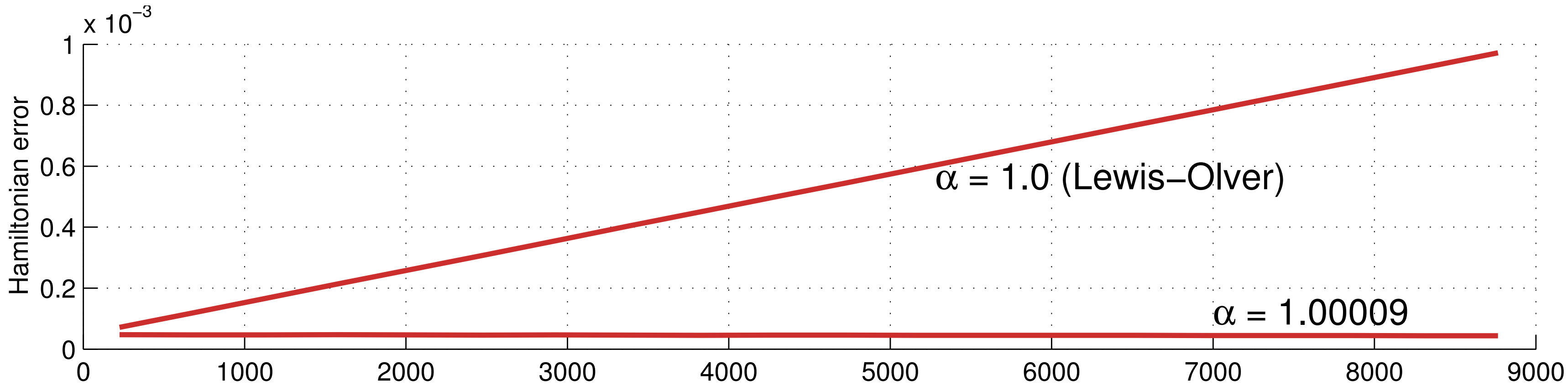
- The second order error is in the direction of the vector field, second order orbit capture.
- Equivalent to the approach in $(*)$.

Results for the rigid body



The isotropy corrected Lie-Euler increases slowly by time.

Remedy: Scale $\sigma(y)$ by a constant α



Acting on R^2 by $SL(2)$

$SL(2)$ is the group of 2×2 matrices with determinant equal to 1. This approach is motivated by the stability properties obtained from isotropy on the rigid body.

The isotropy subalgebra at $y = (u, v)$ in $\mathbf{R}^2$ is

$$\zeta(y) = \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}$$

Equations in R^2

We have $f: \mathbf{R}^2 \to \mathfrak{sl}(2)$ for the following equations:

Lotka-Volterra

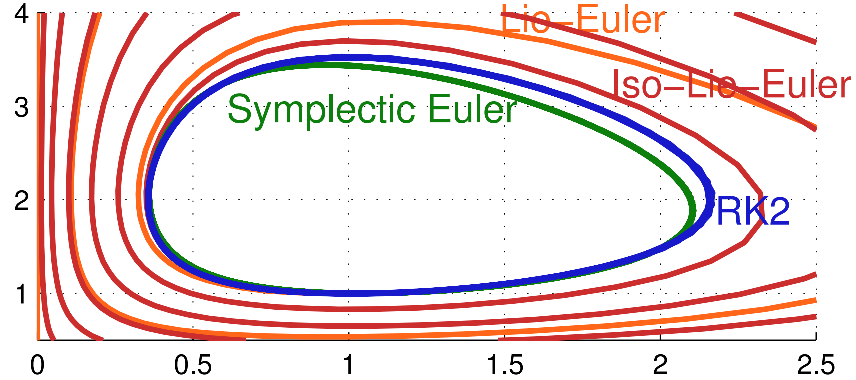
$$\left\{\begin{aligned} \dot u &= u(v-2) \\ \dot v &= v(1-u) \end{aligned}\right\} \quad \Rightarrow \quad f(y) = \begin{pmatrix} u-1 & -\frac{u(u-v+1)}{v} \\ 0 & 1-u \end{pmatrix}$$

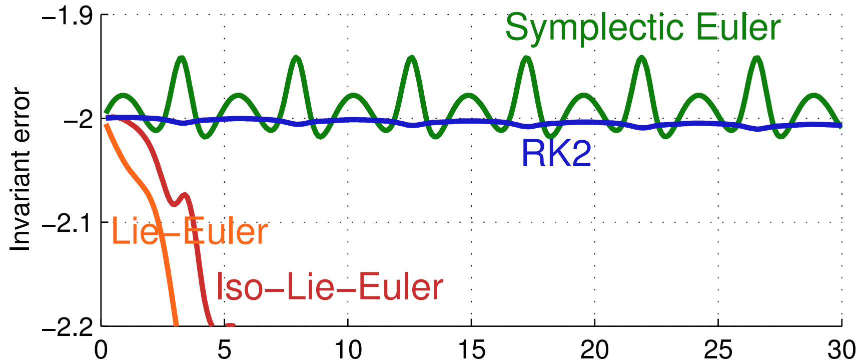
Duffing oscillator

$$\left\{\begin{aligned} \dot u &= v \\ \dot v &= u - u^3 \end{aligned}\right\} \quad \Rightarrow \quad f(y) = \begin{pmatrix} 0 & 1 \\ 1-u^2 & 0 \end{pmatrix}$$

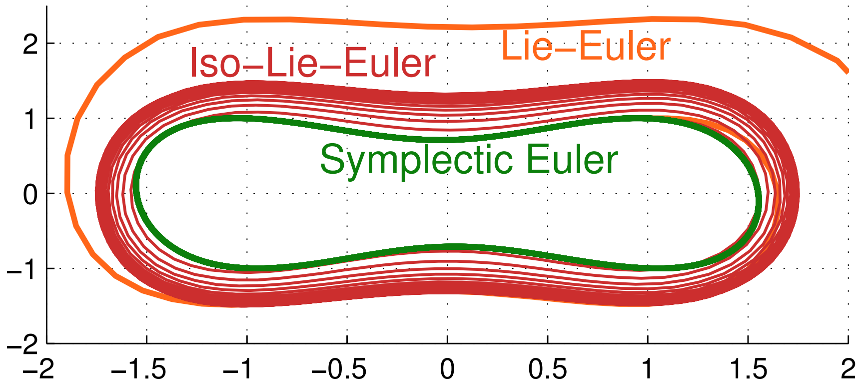
Results on $SL(2)$, $\alpha = 1$

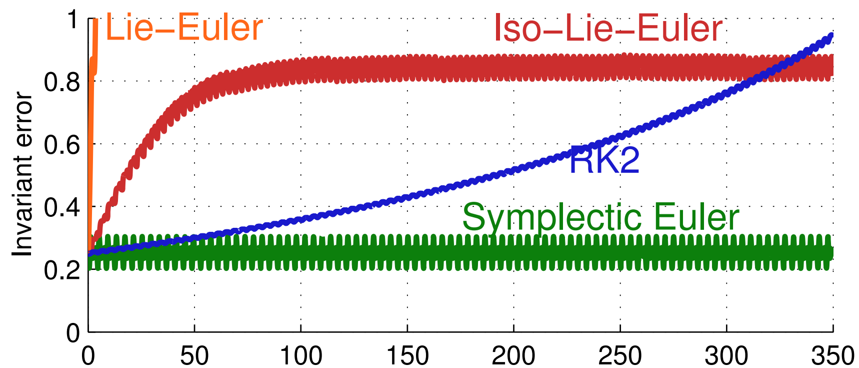
Lotka-Volterra





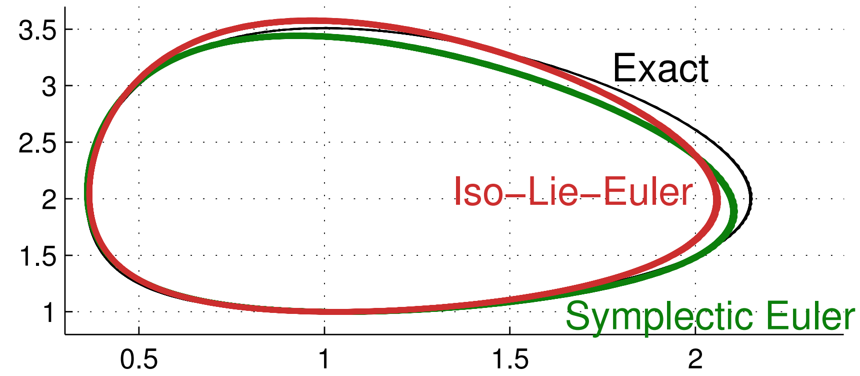
Duffing oscillator

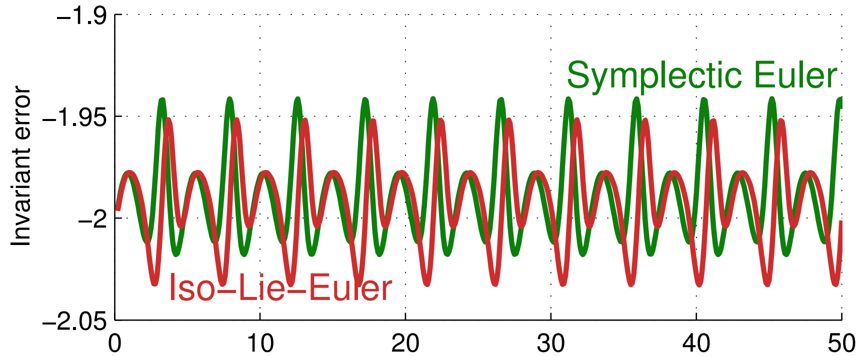




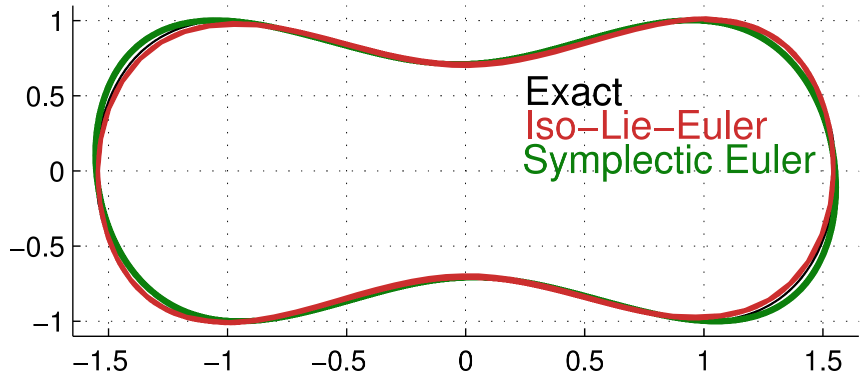
Results on $SL(2)$, $\alpha \neq 1$

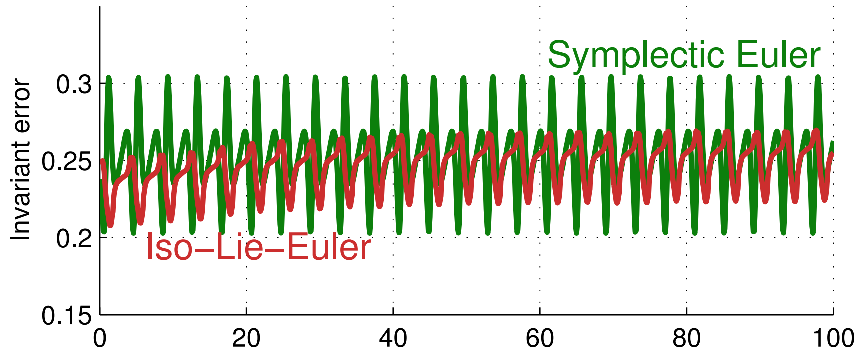
Lotka-Volterra ($\alpha = 1.84$)





Duffing oscillator ($\alpha = 1.17$)





The constant α

Lotka-Volterra	$\alpha = 1.84$
Duffing	$\alpha = 1.17$
Rigid body	$\alpha = 1.00009$

- Is $\alpha = 1.00009$ for rigid body pure luck?
- Dependent on time-step h, f and σ.
- Currently found by trial and error.

The end

Conclusions:

- Taking advantage of isotropy for RKMK methods looks promising in order to attain good global behavior.
- The role of α not yet understood.

Thank you for your attention!

Full diploma thesis available at
http://www.math.ntnu.no/˜berland/thesis